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Advanced Statistical Physics - Problem Set 11

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Due Date: Tuesday, June 26, 09:15 a.m., mailbox inside ITP

Internet: Advanced Statistical Physics exercises

16. Hartree approximation

2+3+3+1 Points

Consider the Landau-Ginzburg Hamiltonian:

$$\beta\mathcal{H} = \int d^d x \left[\frac{t}{2}\boldsymbol{m}^2 + \frac{K}{2}(\nabla \boldsymbol{m})^2 + u(\boldsymbol{m}^2)^2 \right],$$

describing an N-component magnetization vector $\boldsymbol{m}(\boldsymbol{x})$. Assume that $t > 0$.

a) Perform a Hubbard-Stratonovich transformation by first multiplying the partition function by

$$\mathbb{1} = \int D\rho(\mathbf{x}) e^{-N^2 \int d^d x \rho(\mathbf{x})^2/2}$$

and performing a shift $\rho \to \rho + \alpha \boldsymbol{m}^2$ and show that with suitably chosen α you obtain a new Hamiltonian

$$\beta\mathcal{H}[m, \rho] = \int d^d x \left[\frac{t + 2N^2\alpha\rho}{2}\boldsymbol{m}^2 + \frac{K}{2}(\nabla \boldsymbol{m})^2 + \frac{N^2\rho^2}{2} \right] .$$

b) We want to find saddle-point equation, where $\rho(\mathbf{x}) = \rho_0$. Therefore, assume that ρ is constant in space and integrate over $\boldsymbol{m}$ so that you will obtain an effective Hamiltonian for ρ_0

$$\beta H_{\text{eff}}(\rho_0) = \frac{N^2\rho_0^2 V}{2} + \frac{N}{2}\sum_{\mathbf{q}} \ln(t + 2N^2\alpha\rho_0 + Kq^2) .$$

c) Use the effective Hamiltonian obtained in (b) to find the saddle-point equation for ρ_0. Notice that in the Hamiltonian obtained in part (a) t has been renormalized so that $t' = t + 2N^2\alpha\rho$. Use the saddle-point equation to find the self-consistency equation for t'

$$t' = t + \frac{4uN}{(2\pi)^d}\int d^d q \frac{1}{t' + Kq^2} .$$

d) Argue why the method used above works well in the limit $N \to \infty$.

17. Hartree self-consistency equation

2+2+2+2+2 Points

According to the previous problem, the parameter t in the Ginzburg-Landau Hamiltonian is renormalized because of the $u(\boldsymbol{m}^2)^2$ term, and the renormalized parameter t' is determined by the self-consistency equation

$$t' = t + \frac{4uN}{(2\pi)^d} \int d^d q \frac{1}{t' + Kq^2} \ .$$

In this problem we analyze some of the consequences of this renormalization.

a) The critical temperature is shifted and it is determined by the value of t, where the renormalized parameter t' becomes zero. Use the self-consistency equation to calculate the new critical temperature.

b) Rewrite the term inside the integral as

$$\frac{1}{t' + Kq^2} = \frac{1}{t' + Kq^2} - \frac{1}{Kq^2} + \frac{1}{Kq^2} \ ,$$

and show that the self-consistency equation can be written as

$$t' = (t - t_c) - \frac{4uN}{(2\pi)^d} \int d^d q \frac{t'}{(t' + Kq^2)(Kq^2)} \ .$$

c) For $2 < d < 4$ the integral is convergent in both limits. Show that in this case, the self-consistency condition can be written as

$$t' = (t - t_c) - C(d) 4uNK^{-d/2}(t')^{d/2-1} \ ,$$

where $C(d)$ is a constant.

d) In order to determine the critical exponents, we need to know how the renormalized parameter t' depends on the reduced temperature $t - t_c$. Introduce a new variable $M^2 = t'/(t - t_c)$ and show that the self-consistency equation can be written as

$$1 = M^2 \left[1 + C(d)N \left(\frac{t_G}{M^2(t - t_c)} \right)^{(4-d)/2} \right] \ ,$$

where t_G is the Ginzburg temperature. Solve the equation in the limits $t - t_c \gg t_G$ and $t - t_c \ll t_G$.

e) Comment on how the critical exponents for the susceptibility γ, correlation length ν, specific heat α, and magnetization β are changed as compared to the results of the Gaussian model in the two limits discussed above.