Institut für Theoretische Physik
Universität Leipzig

Prof. Dr. B. Rosenow
M. Thamm

Advanced Quantum Mechanics - Problem Set 12

Winter Term 2023/24

Due Date: Hand in solutions to problems marked with * as a single pdf file using Moodle before the lecture on **Thursday, 18.01.2024, 15:15**. The problem set will be discussed in the tutorials on Monday 22.01.2024 and Wednesday 24.01.2024.

Website: https://home.uni-leipzig.de/stp/Quantum_Mechanics_2_WS2324.html

Moodle: https://moodle2.uni-leipzig.de/course/view.php?id=45746

*1. Emission from an atom

1+4+3+4+3 Points

In this problem we will consider a single-electron (the extension to multi-electron atoms is straightforward but we will consider the single-electron case for simplicity) atom with Hamiltonian

$$\hat{H}_0 = \frac{\hat{\boldsymbol{p}}^2}{2m} + V(\boldsymbol{r}),$$

subject to an external electromagnetic field with vector potential $\boldsymbol{A}(\boldsymbol{r}, t)$.

(a) Convince yourself that the Hamiltonian of the atom is given by $\hat{H}_0 + \hat{H}_{\text{para}} + \hat{H}_{\text{dia}}$ with $\hat{H}_{\text{para}} = \frac{e}{m}\boldsymbol{A} \cdot \hat{\boldsymbol{p}}$ denoting the paramagnetic term and $\hat{H}_{\text{dia}} = \frac{1}{2m}(e\boldsymbol{A})^2$ denoting the diamagnetic term.

(b) In Problem Set 10 we found that the Hamiltonian of the electromagnetic field can be written as

$$\hat{H}_{\text{rad}} = \sum_{\boldsymbol{k},\lambda=\pm} \hbar\omega_{\boldsymbol{k}} \left(a^{\dagger}_{\boldsymbol{k},\lambda} a_{\boldsymbol{k},\lambda} + \frac{1}{2} \right),$$

with $\omega_{\boldsymbol{k}} = c|\boldsymbol{k}|$ and where $a^{\dagger}_{\boldsymbol{k},\lambda}$ and $a_{\boldsymbol{k},\lambda}$ create and annihilate photons with wavevector $\boldsymbol{k}$ and polarization λ. These operators satisfy the usual commutation relations of ladder operators and act on a state with $n_{\boldsymbol{k}\lambda}$ photons as $a_{\boldsymbol{k}\lambda}|n_{\boldsymbol{k}\lambda}\rangle = \sqrt{n_{\boldsymbol{k}\lambda}}|n_{\boldsymbol{k}\lambda} - 1\rangle$ and $a^{\dagger}_{\boldsymbol{k}\lambda}|n_{\boldsymbol{k}\lambda}\rangle = \sqrt{n_{\boldsymbol{k}\lambda} + 1}|n_{\boldsymbol{k}\lambda} + 1\rangle$. Moreover if the system is confined to a volume V, the vector potential can be expanded as

$$\boldsymbol{A}(\boldsymbol{r}, t) = \sum_{\boldsymbol{k},\lambda=\pm} \sqrt{\frac{\hbar}{2\epsilon_0\omega_{\boldsymbol{k}} V}} \left(\hat{\boldsymbol{e}}_{\boldsymbol{k}\lambda} a_{\boldsymbol{k}\lambda} e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_{\boldsymbol{k}} t)} + \hat{\boldsymbol{e}}^*_{\boldsymbol{k}\lambda} a^{\dagger}_{\boldsymbol{k}\lambda} e^{-i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_{\boldsymbol{k}} t)} \right).$$

The eigenstates of the combined system of atom and electromagnetic field can then be written as $|nlm\rangle \otimes |n_{\boldsymbol{k}\lambda}\rangle$, where the first part refers to the atomic part of the Hamiltonian and the last part to the radiation part of the Hamiltonian.

Treating the paramagnetic part of the Hamiltonian as a perturbation, show that the probability of a transition between a state $|i\rangle \otimes |n_{\boldsymbol{k}\lambda}\rangle$ and a state $|f\rangle \otimes |n_{\boldsymbol{k}\lambda}+1\rangle$ is given by

$$\Gamma_{if} = \frac{2\pi}{\hbar} \left| \langle f| \frac{e}{m} \sqrt{\frac{\hbar(n_{\boldsymbol{k}\lambda}+1)}{2\epsilon_0 \omega_{\boldsymbol{k}} V}} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \hat{\boldsymbol{e}}^*_{\boldsymbol{k}\lambda} \cdot \hat{\boldsymbol{p}} |i\rangle \right|^2 \delta(E_i - E_f - \hbar\omega_{\boldsymbol{k}}).$$

Hint: You may find it helpful to start from Fermi's golden rule which you may use without proof.

(c) The expression in part (b) is rather complicated. To simplify the matrix element, expand the exponential to zeroth order in $\boldsymbol{k}\cdot\boldsymbol{r}$ (this is valid for $Z\alpha \ll 1$ (why?)) and use the identity $\hat{\boldsymbol{p}} = \frac{im}{\hbar}[\hat{H}_0, \hat{\boldsymbol{r}}]$, to show that

$$\Gamma_{if} = \frac{\pi\omega_{\boldsymbol{k}}(n_{\boldsymbol{k}\lambda}+1)}{\epsilon_0 V} |\langle f|\hat{\boldsymbol{e}}_{\boldsymbol{k}\lambda}\cdot \boldsymbol{d}|i\rangle|^2 \delta(E_i - E_f - \hbar\omega_{\boldsymbol{k}}),$$

where $\boldsymbol{d} = -e\boldsymbol{r}$ is the dipole moment of an electron. This approximation is known as the dipole approximation.

(d) The probability calculated above only tells us about the scattering of photons with the particular momentum $\boldsymbol{k}$. To get the total probability of scattering photons with polarization λ into a solid angle $d\Omega$ we have to perform the sum $d\Gamma = \sum_{\boldsymbol{k}} \Gamma_{if}$. By converting the sum into an integral, assuming that $n_{\boldsymbol{k}\lambda} = n_\lambda(k)$ is isotropic in $\boldsymbol{k}$, and using the dipole approximation show that the total probability per unit solid angle is given by

$$\frac{d\Gamma}{d\Omega} = \frac{1}{4\pi\epsilon_0} \frac{\omega^3(n_\lambda(\omega/c)+1))}{2\pi\hbar c^3} |\langle f|\hat{\boldsymbol{e}}_{\boldsymbol{k}\lambda}\cdot \boldsymbol{d}|i\rangle|^2.$$

Here $\hbar\omega = E_i - E_f$.

(e) Show that the angular average of $|\langle f|\hat{\boldsymbol{e}}_{\boldsymbol{k}\lambda}\cdot \boldsymbol{d}|i\rangle|^2$ is given by $d_{if}^2/3$, with $d_{if}^2 = |\langle f|ex|i\rangle|^2 + |\langle f|ey|i\rangle|^2 + |\langle f|ez|i\rangle|^2$ and thus derive the total emission probability.

Hint: Choose coordinates such that $\boldsymbol{k}$ points in the z-direction and $\mathbf{d}_{if} = \langle f|\boldsymbol{d}|i\rangle = (d_{if}\sin\theta, 0, d_{if}\cos\theta)$. Finally use the Coulomb gauge to derive a condition on $\hat{\boldsymbol{e}}_{\boldsymbol{k}\lambda}$.

2. Emission in Hydrogen *5 Points*

Using the dipole approximatin discussed in the previous problem, calculate the probability of a transition from a 2P state to a 1S state in Hydrogen.

Hint: You can save yourself some work by noticing that only the matrix elements $\langle 1,0,0|x|2,1,\pm1\rangle$, $\langle 1,0,0|y|2,1,\pm1\rangle$, and $\langle 1,0,0|z|2,1,0\rangle$ are non-zero.