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# Advanced Quantum Mechanics - Problem Set 11

*Winter Term 2023/24*

**Due Date:** Hand in solutions to problems marked with * as a single pdf file using Moodle before the lecture on **Thursday, 11.01.2024, 15:15**. The problem set will be discussed in the tutorials on Monday 15.01.2024 and Wednesday 17.01.2024.

**Moodle:** https://moodle2.uni-leipzig.de/course/view.php?id=45746

**Website:** https://home.uni-leipzig.de/stp/Quantum_Mechanics_2_WS2324.html

## *1. Quantisation of the Radiation Field

*2+3+3 Points*

In the absence of charges, and in the Coulomb gauge $\nabla \cdot \boldsymbol{A} = 0$, the electromagnetic field is described by the Lagrangian

$$L(t) = \frac{1}{2} \int_\Omega d^3x \left[ \epsilon_0 \left( \partial_t \boldsymbol{A} \right)^2 + \frac{1}{\mu_0} \boldsymbol{A} \cdot \nabla^2 \boldsymbol{A} \right].$$

Here $\epsilon_0$ denotes the vacuum dielectric constant, $\mu_0$ is the vacuum permeability, and $\Omega$ is a cuboid with extensions $L_x$, $L_y$, and $L_z$. Note that the speed of light is $c = 1/\sqrt{\epsilon_0 \mu_0}$.

(a) Write down the Lagrange equation for $\boldsymbol{A}$.

(b) Find eigenfunctions $\boldsymbol{A}_{\boldsymbol{k}}$ and eigenvalues $\omega_{\boldsymbol{k}}^2$ of the equation

$$-\nabla^2 \boldsymbol{A}(\boldsymbol{x}) = \frac{\omega_{\boldsymbol{k}}^2}{c^2} \boldsymbol{A}(\boldsymbol{x}),$$

by using periodic boundary conditions. It may be useful to introduce, for each $\boldsymbol{k}$, a set of orthonormal vectors $\{\hat{\boldsymbol{\xi}}_{\boldsymbol{k},1}, \hat{\boldsymbol{\xi}}_{\boldsymbol{k},2}\}$ which are both perpendicular to $\boldsymbol{k}$. The time-dependent solution then has a series expansion

$$\boldsymbol{A}(\boldsymbol{x}, t) = \frac{1}{\sqrt{\Omega}} \sum_{\boldsymbol{k},j} \alpha_{\boldsymbol{k},j}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \hat{\boldsymbol{\xi}}_{\boldsymbol{k},j}.$$

Insert this series expansion in the Lagrangian, and find the momenta

$$\pi_{\boldsymbol{k},i} = \frac{\partial L}{\partial \dot{\alpha}_{\boldsymbol{k},i}},$$

canonically conjugate to the coordinates $\alpha_{\boldsymbol{k},i}$. Use the Legendre transform $H = \sum_{\boldsymbol{k},i} \pi_{\boldsymbol{k},i} \dot{\alpha}_{\boldsymbol{k},i} - L(\pi_{\boldsymbol{k},i}, \alpha_{\boldsymbol{k},i})$ to obtain the Hamiltonian.

*Hint: The first equation can be obtained from the Euler-Lagrange equation in (a) by using that $\boldsymbol{A}(\boldsymbol{x}, t) = \mathrm{e}^{-i\omega_{\boldsymbol{k}} t} \boldsymbol{A}(\boldsymbol{x})$. Here, assume that $\boldsymbol{A}(\boldsymbol{x})$ is real. Using this it can be shown that $\alpha_{-\boldsymbol{k},j} = \alpha_{\boldsymbol{k},j}^\dagger$.*

(c) The classical Hamiltonian $H(\{\pi_{\boldsymbol{k},i}, \alpha_{\boldsymbol{k},i}\})$ can be quantised by imposing canonical commutation relations

$$[\alpha_{\boldsymbol{k},i}, \alpha_{\boldsymbol{q},j}] = 0, \quad [\pi_{\boldsymbol{k},i}, \pi_{\boldsymbol{q},j}] = 0, \quad [\alpha_{\boldsymbol{k},i}, \pi_{\boldsymbol{q},j}] = i\hbar\delta_{\boldsymbol{k},\boldsymbol{q}}\delta_{i,j},$$

on the coordinates $\alpha_{\boldsymbol{k},i}$ and their canonically conjugate momenta $\pi_{\boldsymbol{k},j}$. In analogy to the one-dimensional harmonic oscillator, we define photon creation and annihilation operators

$$a^\dagger_{\boldsymbol{k},j} = \sqrt{\frac{\epsilon_0\omega_{\boldsymbol{k}}}{2\hbar}}\left(\alpha_{-\boldsymbol{k},j} - \frac{i}{\epsilon_0\omega_{\boldsymbol{k}}}\pi_{\boldsymbol{k},j}\right), \quad a_{\boldsymbol{k},j} = \sqrt{\frac{\epsilon_0\omega_{\boldsymbol{k}}}{2\hbar}}\left(\alpha_{\boldsymbol{k},j} + \frac{i}{\epsilon_0\omega_{\boldsymbol{k}}}\pi_{-\boldsymbol{k},j}\right).$$

Show that $a_{\boldsymbol{k},j}$ and $a^\dagger_{\boldsymbol{k},j}$ obey the commutation relations of harmonic oscillator ladder operators, and express the Hamiltonian in terms of $a_{\boldsymbol{k},j}$ and $a^\dagger_{\boldsymbol{k},j}$.

## 2. Coulomb and Exchange Integrals for Helium *6+3 Points*

The energy of excited states in helium can be shown to be, to leading order in perturbation theory, given by

$$E_{nl,\pm} = -\frac{Z^2}{2}\left(1 + \frac{1}{n}\right) + J_{nl} \pm K_{nl},$$

where the Coulomb- and exchange integrals for helium are given by

$$J_{nl} = \frac{e^2}{4\pi\epsilon_0}\langle u_{100}(\mathbf{r_1})u_{nlm}(\mathbf{r_2})|\frac{1}{|\mathbf{r_1} - \mathbf{r_2}|}|u_{100}(\mathbf{r_1})u_{nlm}(\mathbf{r_2})\rangle,$$

and

$$K_{nl} = \frac{e^2}{4\pi\epsilon_0}\langle u_{100}(\mathbf{r_1})u_{nlm}(\mathbf{r_2})|\frac{1}{|\mathbf{r_1} - \mathbf{r_2}|}|u_{nlm}(\mathbf{r_1})u_{100}(\mathbf{r_2})\rangle,$$

respectively. Here $u_{nlm}$ is the hydrogen wave-function with $Z = 2$.

(a) Calculate the Coulomb and exchange integrals for $n = 2, l = 0, 1$.

*Hint: Express $1/|\mathbf{r_1} - \mathbf{r_2}|$ as a sum over spherical harmonics and use orthogonality of these to perform the angular integrals.*

(b) Make a sketch of the energy levels $E_{nl,\pm}$ of the terms $^1$S, $^3$S, $^1$P, and $^3$P.

*Hint: Recall that the superscript denotes the spin and is given by $2S + 1$ whilst the letter denotes the total orbital angular momentum $L = L_1 + L_2$ ($L = 0$ for S, $L = 1$ for P).*