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Advanced Quantum Mechanics - Problem Set 10

Winter Term 2023/24

Due Date: Hand in solutions to problems marked with * as a single pdf file using Moodle before the lecture on **Thursday, 04.01.2024, 15:15**. The problem set will be discussed in the tutorials on Monday 08.01.2024 and Wednesday 10.01.2024.

Website: https://home.uni-leipzig.de/stp/Quantum_Mechanics_2_WS2324.html

Moodle: https://moodle2.uni-leipzig.de/course/view.php?id=45746

*1. Spin 1 system

3+2 Points

The Hamiltonian for a spin 1 system is given by

$$\hat{H} = A\hat{S}_z^2 + B(\hat{S}_x^2 - \hat{S}_y^2),$$

where the $\hat{S}_i$ are spin operators and A, B are real constants.

(a) Find the normalized energy eigenstates and eigenvalues.

(b) Is the Hamiltonian invariant under time reversal? How do the normalized eigenstates you calculated in part (a) transform under time reversal?

2. Addition of angular momenta

2+1+3+1.5+2.5 Points

Consider the operators of orbital angular momentum $\hat{\boldsymbol{L}}$ and spin $\hat{\boldsymbol{S}}$ with angular momentum eigenstates $|l, m_l\rangle$ and $|s, m_s\rangle$, respectively, as well as the total angular momentum operator $\hat{\boldsymbol{J}} = \hat{\boldsymbol{L}} + \hat{\boldsymbol{S}}$ with angular momentum eigenstates $|j, m_j\rangle$.

(a) Given $l = 1$ and $s = 1/2$, write down all possible combinations of j and m_j.

(b) The state $|j = 3/2, m_j = 3/2\rangle$ can immediately be expressed in terms of the $|l = 1, m_l\rangle|s = 1/2, m_s\rangle$ state vectors:

$$|j = 3/2, m_j = 3/2\rangle = |l = 1, m_l = 1\rangle|s = 1/2, m_s = 1/2\rangle.$$

Write down a corresponding expression for the state $|j = 3/2, m_j = -3/2\rangle$.

(c) We can construct the state $|j = 3/2, m_j = 1/2\rangle$, expressed in terms of the $|l = 1, m_l\rangle|s = 1/2, m_s\rangle$

state vectors, by application of $J_- = L_- + S_-$ to $|j = 3/2, m_j = 3/2\rangle$:

$$
\begin{aligned}
& J_-|j = 3/2, m_j = 3/2\rangle = (L_- + S_-)\,|l = 1, m_l = 1\rangle|s = 1/2, m_s = 1/2\rangle \\
\Leftrightarrow \quad & \sqrt{3}|j = 3/2, m_j = 1/2\rangle = \sqrt{2}|l = 1, m_l = 0\rangle|s = 1/2, m_s = 1/2\rangle \\
& \qquad\qquad + |l = 1, m_l = 1\rangle|s = 1/2, m_s = -1/2\rangle \\
\Rightarrow \quad & |j = 3/2, m_j = 1/2\rangle = \sqrt{\frac{2}{3}}|l = 1, m_l = 0\rangle|s = 1/2, m_s = 1/2\rangle \\
& \qquad\qquad + \sqrt{\frac{1}{3}}|l = 1, m_l = 1\rangle|s = 1/2, m_s = -1/2\rangle.
\end{aligned}
$$

Repeat this procedure (that is, apply J_- to $|j = 3/2, m_j = 1/2\rangle$ given above), to express the state $|j = 3/2, m_j = -1/2\rangle$ in terms of the $|l = 1, m_l\rangle|s = 1/2, m_s\rangle$ state vectors.

(d) Use that $|j = 3/2, m_j = 1/2\rangle$ (given above) and $|j = 1/2, m_j = 1/2\rangle$ are orthonormal and contain the same $|l = 1, m_l\rangle|s = 1/2, m_s\rangle$ states, to express $|j = 1/2, m_j = 1/2\rangle$ in terms of the $|l = 1, m_l\rangle|s = 1/2, m_s\rangle$ states.

(e) Apply J_- to your result for $|j = 1/2, m_j = 1/2\rangle$ obtained in (d), to construct the state $|j = 1/2, m_j = -1/2\rangle$ expressed in terms of the $|l = 1, m_l\rangle|s = 1/2, m_s\rangle$ states.

Hint: The ladder operator J_- acts on the state $|j, m_j\rangle$ in the following way: $J_-|j, m_j\rangle = \hbar\sqrt{j(j+1) - m_j(m_j - 1)}|j, m_j - 1\rangle$. Analogous relations hold for the operators L_- and S_-.