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Advanced Quantum Mechanics - Problem Set 7

Winter Term 2023/24

Due Date: Hand in solutions to problems marked with * as a single pdf file using Moodle before the lecture on **Thursday, 30.11.2023, 15:15**. The problem set will be discussed in the tutorials on Monday 04.12.2023 and Wednesday 06.12.2023.

Website: https://home.uni-leipzig.de/stp/Quantum_Mechanics_2_WS2324.html

Moodle: https://moodle2.uni-leipzig.de/course/view.php?id=45746

1. Representations of γ matrices *2+1+2 Points*

The γ matrices can be written as

$$\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \qquad i = 1, 2, 3$$
$$\gamma_0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix},$$

where σ_i denotes a Pauli matrix and $\mathbb{1}_n$ the $n \times n$ unit matrix. Consider the metric $\eta = \mathrm{diag}(1, -1, -1, -1)$.

(a) Show that the γ matrices satisfy the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\mathbb{1}_4$, $\mu, \nu \in \{0, 1, 2, 3\}$.

(b) A different representation is the Weyl representation where

$$\gamma_0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}.$$

Show that these still satisfy the Clifford algebra.

(c) Using only the Clifford algebra and properties of the trace show that $\mathrm{tr}(\gamma_\mu) = 0$, $\mathrm{tr}(\gamma_\mu\gamma_\nu) = 4\eta_{\mu\nu}$, and $\mathrm{tr}(\gamma_\mu\gamma_\nu\gamma_\rho) = 0$.

*2. Continuity equation for the Dirac equation *5 Points*

Prove the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0,$$

with

$$\boldsymbol{j} = \Psi^\dagger \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \Psi,$$

and $\rho = \Psi^\dagger\Psi$ for all solutions Ψ of the Dirac equation.

3. Free particle solutions of the Dirac equation *3 Points*

Calculate the eigenvalues of the free-particle Dirac equation

$$\begin{pmatrix} m & 0 & p & 0 \\ 0 & m & 0 & -p \\ p & 0 & -m & 0 \\ 0 & -p & 0 & -m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = E \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}.$$

4. Commutators of Dirac matrices *2+2 Points*

Consider the Dirac matrices

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix},$$
$$\beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix},$$

where $\boldsymbol{\sigma}$ is the vector of Pauli matrices and $\mathbb{1}_2$ is the 2-dimensional unit matrix. Define also

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}.$$

Show that (i) $\beta\Sigma_i = \Sigma_i\beta$, and that (ii) $[\alpha_i, \Sigma_j] = 2i\epsilon_{ijk}\alpha_k$.