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Advanced Quantum Mechanics - Problem Set 0

Winter Term 2023/24

Due Date: Hand in solutions to problems marked with * to mailbox 39 with label "Advanced Quantum Mechanics" inside ITP room 105b before the lecture on **Thursday, 12.10.2023, 15:15**. The problem set will be discussed in the tutorials on Monday 16.10.2023 and Wednesday 18.10.2023.

Internet: https://home.uni-leipzig.de/stp/Quantum_Mechanics_2_WS2324.html

The aim of the problem set is to familiarize yourself with Dirac notation.

*1. Two-level system *3 Points*

Consider the Hamiltonian of a two-level system

$$\hat{H} = a\big(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|\big),$$

where $a > 0$ has the dimension of an energy. Calculate the energy eigenvalues and eigenstates with respect to the orthonormal basis $\{|1\rangle, |2\rangle\}$.

2. Unitary transformation *1+2 Points*

Consider the unitary transformation $|\psi'\rangle = \hat{U}|\psi\rangle$.

(a) Show that the operator $\hat{A}$ has to be transformed as $\hat{A}' = \hat{U}\hat{A}\hat{U}^\dagger$

(b) Show that with the above definitions the following properties of the operators are conserved in the transformation:

(i) linearity and hermiticity

(ii) commutation relations

(iii) the eigenvalue spectrum

(iv) the algebraic relations $\hat{F} = \hat{K} + \hat{M}$ and $\hat{F} = \hat{K}\hat{M}$

3. Momentum representation

2+2 Points

Let $|\alpha\rangle$ and $|\beta\rangle$ be arbitrary ket-vectors. Use the normalization $\langle p|p'\rangle = \delta(p-p')$ and completeness relation $\int dx|x\rangle\langle x| = \hat{\mathbb{1}}$ to obtain an expression for $\langle x|p\rangle$. Show then explicitly

(a) $\langle p|\hat{x}|\alpha\rangle = i\hbar\frac{\partial}{\partial p}\psi_\alpha(p)$,

(b) $\langle\beta|\hat{x}|\alpha\rangle = \int dp\ \psi_\beta^*(p)i\hbar\frac{\partial}{\partial p}\psi_\alpha(p)$.

Here $\psi_\alpha(p) = \langle p|\alpha\rangle$ and $\psi_\beta(p) = \langle p|\beta\rangle$ are one-dimensional wave functions in momentum representation and $\hat{x}$ is the position operator.

Hint: Use the Fourier-representation of the delta function: $\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \exp\left[i\left(x-x'\right)y\right]$.

4. Change of representation

4 Points

Let us denote the eigenstate of the position operator $\hat{x}$ with eigenvalue x as $|x\rangle$, the eigenstate of the momentum operator $\hat{p}$ with eigenvalue p as $|p\rangle$ and the eigenstate of the Hamilton operator $\hat{H} = \frac{\hat{p}^2}{2m}$ with energy E as $|E\rangle$. Consider a particle in the state $|\Psi\rangle$ which in the momentum representation is given by $\langle p|\Psi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp(-ix_0\frac{p}{\hbar})$.

(a) Calculate $\langle x|\Psi\rangle$. How can the state $|\Psi\rangle$ therefore be physically interpreted?

(b) Use the eigenvalue equation for $\hat{H}$ and the matrix elements $\langle x|\hat{H}|x'\rangle = -\frac{d^2}{dx^2}\delta(x-x')\frac{\hbar^2}{2m}$ to derive a differential equation for $\Psi_E(x) = \langle x|E\rangle$ and obtain $\Psi_E(x)$.

(c) Use your results obtained in (b) to calculate $\langle p|E\rangle$ and express the eigenvalues E in terms of the eigenvalues p.