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Advanced Quantum Mechanics - Problem Set 13

Winter Term 2022/23

Due Date: Hand in solutions to problems marked with * to mailbox 39 with label "Advanced Quantum Mechanics" inside ITP room 105b before the lecture on **Friday, 27.01.2023, 09:15**. The problem set will be discussed in the tutorials on Monday 30.01.2023 and Wednesday 01.02.2023.

*1. Unit cell in the presence of a magnetic field *2+5+3 Points*

Recall that the operator $\hat{T}_{\boldsymbol{a}} = e^{\frac{i}{\hbar}\boldsymbol{a}\cdot\hat{\boldsymbol{p}}}$ is the generator of translations. For a Hamiltonian with lattice translation symmetry, these operators commute with the Hamiltonian. In a magnetic field this is no longer the case since the vector potential is not translationally invariant. In this problem we will consider a two-dimensional electron gas in the presence of a magnetic field in the z-direction $\boldsymbol{B} = (0, 0, B)$. The Hamiltonian can be written as

$$\hat{H} = \frac{(\hat{\boldsymbol{p}} - e\boldsymbol{A}(\boldsymbol{r}))^2}{2m} + V(\boldsymbol{r}),$$

where $V(\boldsymbol{r})$ is the periodic lattice potential, i.e. $V(\boldsymbol{r} + \boldsymbol{a}) = V(\boldsymbol{r})$ for lattice vectors $\boldsymbol{a}$. For this problem we use the symmetric gauge $\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{2}(-By, Bx, 0)$.

(a) Show that the translation operator

$$\hat{\mathcal{T}}_{\boldsymbol{a}} = \exp\left\{\frac{i}{\hbar}\,\boldsymbol{a}\cdot[\hat{\boldsymbol{p}} + e\boldsymbol{A}(\boldsymbol{r})]\right\}$$

commutes with the Hamiltonian. This translation operator is called a magnetic translation operator.

(b) Show that

$$\hat{\mathcal{T}}_{\boldsymbol{a}}\hat{\mathcal{T}}_{\boldsymbol{b}} = \exp\left[\frac{i}{l_0^2}(\boldsymbol{a}\times\boldsymbol{b})\cdot\hat{\boldsymbol{e}}_z\right]\hat{\mathcal{T}}_{\boldsymbol{b}}\hat{\mathcal{T}}_{\boldsymbol{a}}.$$

Here $l_0 = \sqrt{\frac{\hbar}{eB}}$ is the magnetic length and $\hat{\boldsymbol{e}}_z$ is a unit vector perpendicular to the plane.

(c) We now want to determine the enlarged unit cell such that the magnetic translation operators commute with each other. Let therefore $n\boldsymbol{a}$ and $m\boldsymbol{b}$ span an enlarged unit cell in the plane. In this case the magnetic translation operators have to commute with each other. Show that this is only possible if the flux $\Phi = \boldsymbol{B}\cdot(\boldsymbol{a}\times\boldsymbol{b})$ satisfies

$$\frac{\Phi}{\Phi_0} = \frac{l}{mn},$$

with l an integer and $\Phi_0 = h/e$.

2. Anyons and the Aharonov-Bohm effect

2+2 Points

Consider a two-dimensional electron gas in the presence of a magnetic field. The conductivity tensor is given by

$$\sigma = \begin{pmatrix} 0 & \sigma_{xy} \\ -\sigma_{xy} & 0 \end{pmatrix},$$

where $\sigma_{xy} = \nu e^2/h$, with $0 < \nu < 1$, is the Hall conductivity.

(a) Suppose now a flux Φ is turned on adiabatically. Using Faraday's law and that the current density is given by $\boldsymbol{J} = \sigma \boldsymbol{E}$, where $\boldsymbol{E}$ is the induced electric field, show that the charge satisfies $\dot{Q} = \sigma_{xy}\dot{\Phi}$. How does the charge change if the flux changes by $\Phi_0 = h/e$?

(b) Now consider the composite object (quasiparticle) of a flux Φ_0 and charge $q = \nu e$. Determine the mutual statistics of these quasi particles. When do these composite objects behave as electrons? What do you get for $\nu = 1/3$ and $\nu = 1/5$? These states have been observed in experiments.

Hint: The exchange of the two quasi particles corresponds to moving one quasi particle by half a circle and performing a translation. This suggests that the wave function acquires a phase, which is half of the Berry phase acquired by a charge $q = \nu e$ moving along a path enclosing a magnetic flux Φ_0 (see problem set 12 task 3). Use this reasoning to obtain the exchange statistics of the composite objects for different values of ν. Quasi particles which acquire a phase different from 0 (bosons) or π (fermions) in the exchange are called anyons.

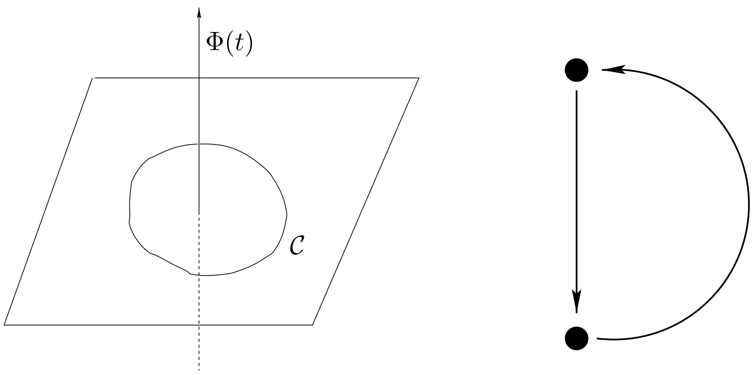

Figure 1: Left: The composite object is made up of a flux enclosed by a path $\mathcal{C}$ and a charge. Right: Illustration of how to exchange two quasi particles.