Institut für Theoretische Physik
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Advanced Quantum Mechanics - Problem Set 12

Winter Term 2022/23

Due Date: Hand in solutions to problems marked with * to mailbox 39 with label "Advanced Quantum Mechanics" inside ITP room 105b before the lecture on **Friday, 20.01.2023, 09:15**. The problem set will be discussed in the tutorials on Monday 23.01.2023 and Wednesday 25.01.2023.

*1. Number operator *4 Points*

Consider an operator $\hat{a}$ which satisfies $\{\hat{a}, \hat{a}^\dagger\} = \hat{a}\hat{a}^\dagger + \hat{a}^\dagger a = 1$ and $\{a, a\} = \{a^\dagger, a^\dagger\} = 0$. Show that the operator $\hat{N} = \hat{a}^\dagger \hat{a}$ has eigenvalues 0 and 1. What would you get if the anti-commutator is replaced by a commutator?

*2. Tight-binding model *2+2+2+2 Points*

In this problem we consider a tight-binding model defined on a one-dimensional lattice with N sites. The Hamiltonian can, in second quantised notation, be written as

$$H = -t \sum_i c_{i+1}^\dagger c_i + \text{h.c.},$$

where the sum is over lattice sites $i \in \mathbb{Z}$, $c_i^\dagger$ and c_i are creation and annihilation operators satisfying $\{c_i, c_j^\dagger\} = \delta_{ij}$, and h.c. stands for hermitian conjugate.

(a) Show that the state

$$|k\rangle = \frac{1}{\sqrt{N}} \sum_j e^{ikj} c_j^\dagger |0\rangle,$$

with $|0\rangle$ denoting a state with no particles, is an eigenstate of the Hamiltonian.

(b) Define now

$$c_k = \frac{1}{\sqrt{N}} \sum_j e^{-ikj} c_j.$$

Show that $\{c_k, c_{k'}^\dagger\} = \delta_{kk'}$.

(c) Show that the inverse transformation is

$$c_j = \frac{1}{\sqrt{N}} \sum_k e^{ikj} c_k.$$

(d) Show that the Hamiltonian can be written in terms of the new operators as

$$H = \sum_k \epsilon(k) c_k^\dagger c_k,$$

where $\epsilon(k)$ is the spectrum.

3. Berry phase and the Aharonov-Bohm effect *2+2+1+3 Points*

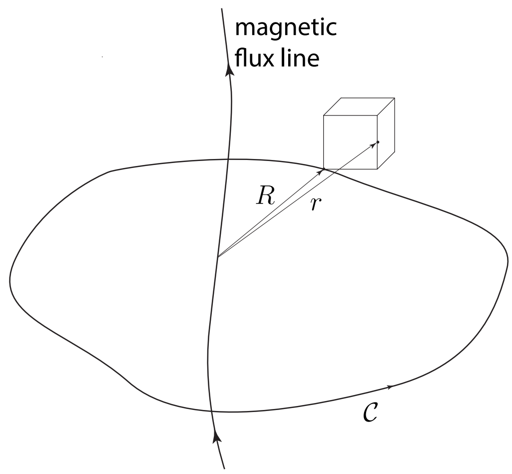

Figure 1: An electron in a box moves around a magnetic flux line. The path of the electron encloses a flux Φ_B.

Consider an electron in a small box moving along a closed loop $\mathcal{C}$, which encloses a magnetic flux Φ_B as shown in Fig. 1. Let $\boldsymbol{R}$ denote the position vector of a point on the box and $\boldsymbol{r}$ the position vector of the electron itself.

(a) Show that if the wave function of the electron in the absence of a magnetic field is $\psi_n(\boldsymbol{r} - \boldsymbol{R})$, then the wave function of the electron in the box at position $\boldsymbol{r}$ is

$$\langle \boldsymbol{r} | n(\boldsymbol{R}) \rangle = \exp\left[\frac{ie}{\hbar} \int_{\boldsymbol{R}}^{\boldsymbol{r}} \boldsymbol{A}(\boldsymbol{r'}) \cdot d\boldsymbol{r'} \right] \psi_n(\boldsymbol{r} - \boldsymbol{R}).$$

Here $\boldsymbol{A}$ denotes the vector potential. Note that this is only true if the magnetic field inside the box is zero. Why?

(b) Show that

$$\langle n(\boldsymbol{R}) | \nabla_{\boldsymbol{R}} | n(\boldsymbol{R}) \rangle = -\frac{ie}{\hbar} \boldsymbol{A}(\boldsymbol{R}).$$

(c) Calculate the geometric phase

$$\gamma_n(\mathcal{C}) = i \oint_{\mathcal{C}} \langle n(\boldsymbol{R}) | \nabla_{\boldsymbol{R}} | n(\boldsymbol{R}) \rangle \cdot d\boldsymbol{R},$$

and comment on your result.

(d) Suppose now an electron moves above or below a very long impenetrable cylinder as shown in the Fig. 2. Inside the cylinder there is a magnetic field parallel to the cylinder axis, taken to be normal to the plane of the figure. Outside the cylinder there is no magnetic field but the particle paths enclose a magnetic flux. Calculate the interference due to the presence of the magnetic flux.

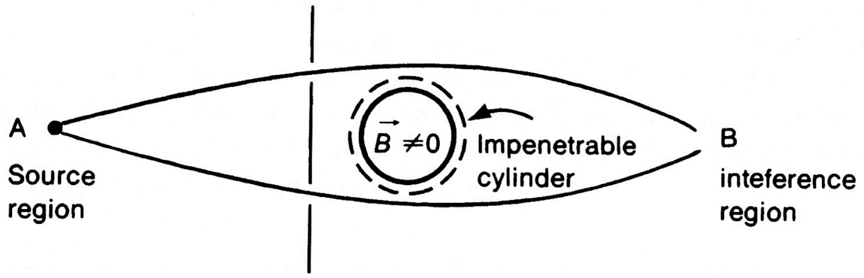

Figure 2: An electron moves either above or below an impenetrable cylinder enclosing a magnetic field parallel to its axis.