Institut für Theoretische Physik
Universität Leipzig

Prof. Dr. B. Rosenow
Dr. C. Drukier
Dr. T. Hyart

Quantum Mechanics 2 - Problem Set 12

Wintersemester 2017/2018

Abgabe: The problem set will be discussed in the tutorials on **Thursday, 18.01.2018, 11:00** (German) and **Friday, 19.01.2018, 13:30** (English).

36. Number operator *4 Punkte*

Consider an operator $\hat{a}$ which satisfies $\{\hat{a}, \hat{a}^\dagger\} = \hat{a}\hat{a}^\dagger + \hat{a}^\dagger a = 1$ and $\{a, a\} = \{a^\dagger, a^\dagger\} = 0$. Show that the operator $\hat{N} = \hat{a}^\dagger \hat{a}$ has eigenvalues 0 and 1. What would you get if the anti-commutator is replaced by a commutator?

37. Tight-binding model *2+2+2+2 Punkte*

In this problem we consider a tight-binding model defined on a one-dimensional lattice with N sites. The Hamiltonian can, in second quantised notation, be written as

$$H = -t \sum_i c_{i+1}^\dagger c_i + \text{h.c.},$$

where the sum is over lattice sites $i \in \mathbb{Z}$, $c_i^\dagger$ and c_i are creation and annihilation operators satisfying $\{c_i, c_j^\dagger\} = \delta_{ij}$, and h.c. stands for hermitian conjugate.

(a) Show that the state

$$|k\rangle = \frac{1}{\sqrt{N}} \sum_j e^{ikj} c_j^\dagger |0\rangle,$$

with $|0\rangle$ denoting a state with no particles, is an eigenstate of the Hamiltonian.

(b) Define now

$$c_k = \frac{1}{\sqrt{N}} \sum_j e^{-ikj} c_j.$$

Show that $\{c_k, c_{k'}^\dagger\} = \delta_{kk'}$.

(c) Show that the inverse transformation is

$$c_j = \frac{1}{\sqrt{N}} \sum_k e^{ikj} c_k.$$

(d) Show that the Hamiltonian can be written in terms of the new operators as

$$H = \sum_k \epsilon(k) c_k^\dagger c_k,$$

where $\epsilon(k)$ is the spectrum.

38. Berry phase and the Aharonov-Bohm effect *2+2+1+3 Punkte*

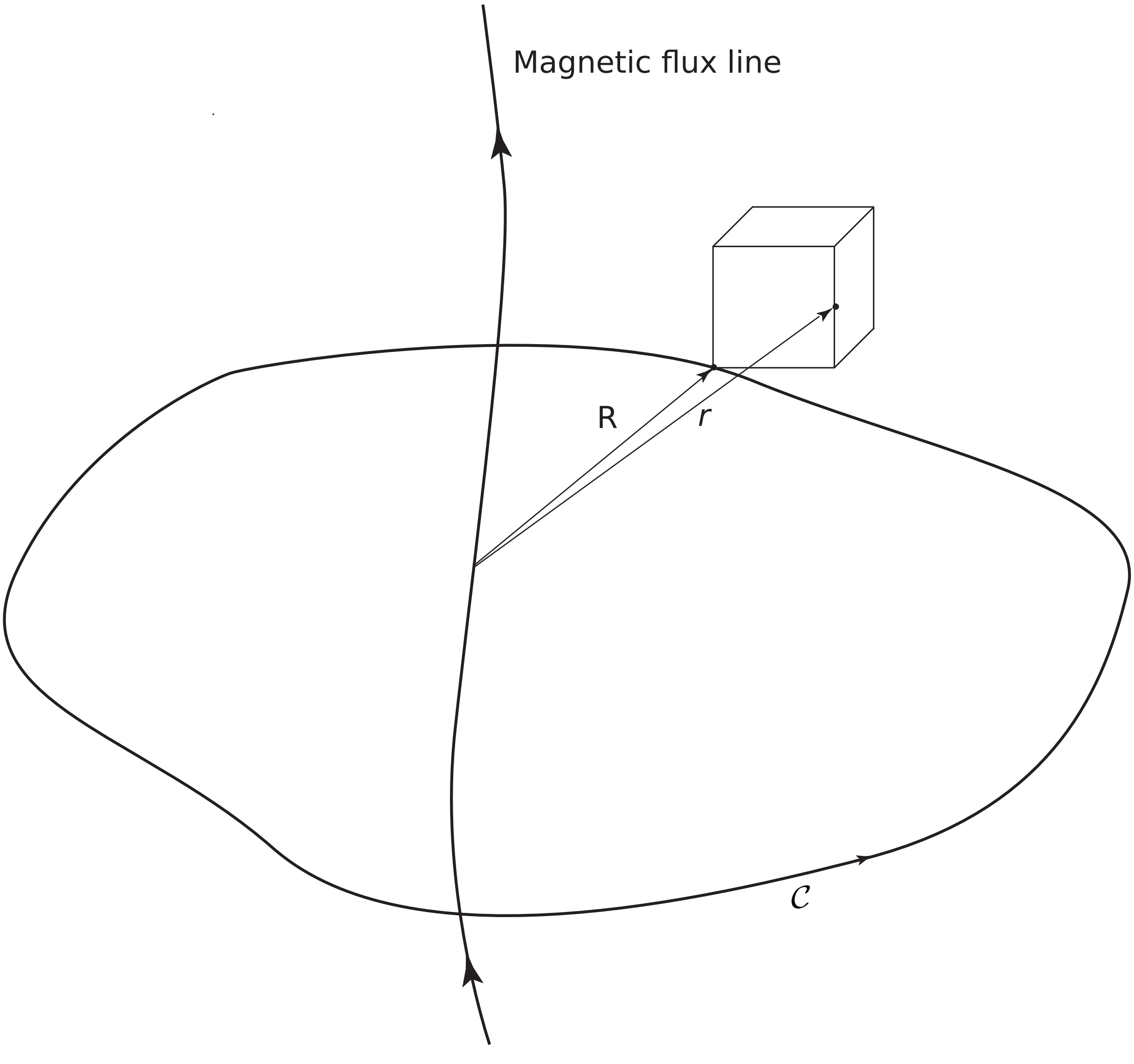

Figure 1: An electron in a box moves around a magnetic flux line. The path of the electron encloses a flux Φ_B.

Consider an electron in a small box moving along a closed loop $\mathcal{C}$, which encloses a magnetic flux Φ_B as shown in Fig. 1. Let $\mathbf{R}$ denote the position vector of a point on the box and $\mathbf{r}$ the position vector of the electron itself.

(a) By denoting the eigenstate in the absence of a magnetic field as $\psi_n(\mathbf{r}-\mathbf{R})$, show that in the presence of the magnetic field the eigenstate is

$$\langle \mathbf{r}|n(\mathbf{R})\rangle = \exp\left[\frac{ie}{\hbar}\int_{\mathbf{R}}^{\mathbf{r}} \mathbf{A}(\mathbf{r}')\cdot d\mathbf{r}'\right]\psi_n(\mathbf{r}-\mathbf{R}).$$

Here $\mathbf{A}$ denotes the vector potential. Notice that this is only true if the magnetic field inside the box is zero. Why?

(b) Show that

$$\langle n(\mathbf{R})|\nabla_{\mathbf{R}}|n(\mathbf{R})\rangle = -\frac{ie}{\hbar}\mathbf{A}(\mathbf{R}).$$

(c) Calculate the geometric phase

$$\gamma_n(\mathcal{C}) = i\oint_{\mathcal{C}} \langle n(\mathbf{R})|\nabla_{\mathbf{R}}|n(\mathbf{R})\rangle \cdot d\mathbf{R},$$

and comment on your result.

(d) Suppose now an electron moves above or below a very long impenetrable cylinder as shown in the Fig. 2. Inside the cylinder there is a magnetic field parallel to the cylinder axis, taken to be normal to the plane of the figure. Outside the cylinder there is no magnetic field but the particle paths enclose a magnetic flux. Calculate the interference due to the presence of the magnetic flux.

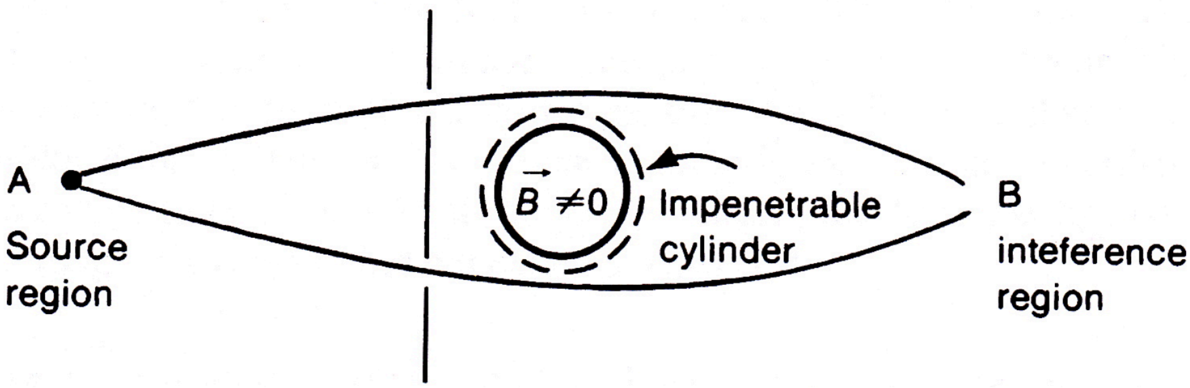

Figure 2: An electron moves either above or below an impenetrable cylinder enclosing a magnetic field parallel to its axis.