Institut für Theoretische Physik
Universität Leipzig

Prof. Dr. B. Rosenow
Dr. C. Drukier
Dr. T. Hyart

Quantum Mechanics 2 - Problem Set 9

Wintersemester 2017/2018

Abgabe: The problem set will be discussed in the tutorials on **Thursday, 14.12.2017, 11:00** (German) and **Friday, 15.12.2017, 13:30** (English).

28. Spin-orbit coupling in Hydrogen

4+2+1 Punkte

The spin-orbit Hamiltonian for Hydrogen is given by

$$H_{\mathrm{SO}} = \frac{e^2}{4\pi\epsilon_0}\frac{1}{m^2c^2r^3}\mathbf{S}\cdot\mathbf{L}.$$

We will treat this Hamiltonian as a perturbation in this problem.

(a) Using the relevant Hydrogen wave-function, calculate the leading order energy correction due to spin-orbit coupling, for $n = 2$, and $l = 1$. Take $s = 1/2$ as the spin of the electron.

(b) Use Kramers' relation

$$\frac{\alpha+1}{n^2}\langle r^\alpha\rangle - (2\alpha+1)a\langle r^{\alpha-1}\rangle + \frac{\alpha}{4}\left[(2l+1)^2-\alpha^2\right]a^2\langle r^{\alpha-2}\rangle = 0,$$

where a is the Bohr radius, to derive a relation between $\langle r^{-2}\rangle$ and $\langle r^{-3}\rangle$.

(c) Calculate the leading order energy correction due to spin-orbit coupling for general n and l. You may use that

$$\langle r^{-2}\rangle = \frac{1}{(l+1/2)n^3a^2}.$$

29. Addition of three angular momenta

3+2 Punkte

Consider three angular momenta with $l_1 = l_2 = l_3 = 1$.

(a) Add the three angular momenta to get a state with total angular momentum $L = 0$.

Hint: First add L_1 and L_2 and then add the resulting angular momentum with L_3. Use the same basis as in the problem 26. Keep only the basis functions that can result to $L = 0$.

(b) Show that this state can be written as a 3×3 determinant and that it therefore is antisymmetric.

30. Quantisation of the Radiation Field

2+3+3 Punkte

In the absence of charges, and in the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, the electromagnetic field is described by the Lagrangian

$$L = \frac{1}{2} \int_{\Omega} d^3x \left[\epsilon_0 \left(\partial_t \mathbf{A}\right)^2 + \frac{1}{\mu_0} \mathbf{A} \nabla^2 \mathbf{A} \right].$$

Here ϵ_0 denotes the vacuum dielectric constant, μ_0 is the vacuum permeability, and Ω is a cuboid with extensions L_x, L_y, and L_z. Note that the speed of light is $c = 1/\sqrt{\epsilon_0 \mu_0}$.

a) Write down the Lagrange equation for $\mathbf{A}$.

b) Find eigenfunctions $\mathbf{A}_\mathbf{k}$ and eigenvalues $\omega_\mathbf{k}^2$ of the equation

$$-\nabla^2 \mathbf{A} = \frac{\omega_\mathbf{k}^2}{c^2} \mathbf{A},$$

by using periodic boundary conditions. It is useful to introduce, for each $\mathbf{k}$, orthonormal vectors $\hat{\xi}_{\mathbf{k},1}$ and $\hat{\xi}_{\mathbf{k},2}$ so that they are perpendicular to $\mathbf{k}$. The time-dependent solution then has a series expansion

$$\mathbf{A}(\mathbf{x}, t) = \frac{1}{\sqrt{\Omega}} \sum_{\mathbf{k},i} \alpha_{\mathbf{k},i}(t) e^{i\mathbf{k}\cdot\mathbf{x}} \hat{\xi}_{\mathbf{k},i}.$$

Insert this series expansion in the Lagrangian, and find the momenta

$$\pi_{\mathbf{k},i} = \frac{\partial L}{\partial \dot{\alpha}_{\mathbf{k},i}},$$

canonically conjugate to the coordinates $\alpha_{\mathbf{k},i}$. Use the Legendre transform $H = \sum_{\mathbf{k},i} \pi_{\mathbf{k},i} \dot{\alpha}_{\mathbf{k},i} - L(\pi_{\mathbf{k},i}, \alpha_{\mathbf{k},i})$ to obtain the Hamiltonian.

c) The classical Hamiltonian $H(\{\pi_{\mathbf{k},i}, \alpha_{\mathbf{k},i}\})$ can be quantised by imposing canonical commutation relations

$$[\alpha_{\mathbf{k},i}, \alpha_{\mathbf{q},j}] = 0, \quad [\pi_{\mathbf{k},i}, \pi_{\mathbf{q},j}] = 0, \quad [\alpha_{\mathbf{k},i}, \pi_{\mathbf{q},j}] = i\hbar \delta_{\mathbf{k},\mathbf{q}} \delta_{i,j},$$

on the coordinates $\alpha_{\mathbf{k},i}$ and their canonically conjugate momenta $\pi_{\mathbf{k},j}$. In analogy to the one-dimensional harmonic oscillator, we define photon creation and annihilation operators

$$a^\dagger_{\mathbf{k},j} = \sqrt{\frac{\epsilon_0 \omega_\mathbf{k}}{2\hbar}} \left(\alpha_{-\mathbf{k},j} - \frac{i}{\epsilon_0 \omega_\mathbf{k}} \pi_{\mathbf{k},j} \right), \quad a_{\mathbf{k},j} = \sqrt{\frac{\epsilon_0 \omega_\mathbf{k}}{2\hbar}} \left(\alpha_{\mathbf{k},j} + \frac{i}{\epsilon_0 \omega_\mathbf{k}} \pi_{-\mathbf{k},j} \right).$$

Show that $a_{\mathbf{k},j}$ and $a^\dagger_{\mathbf{k},j}$ obey the commutation relations of harmonic oscillator ladder operators, and express the Hamiltonian in terms of $a_{\mathbf{k},j}$ and $a^\dagger_{\mathbf{k},j}$.