
Quantum Mechanics 2 - Problem Set 1

Wintersemester 2017/2018

Abgabe: The problem set will be discussed in the tutorials on **Thursday, 19.10.2017, 11:00** (German) and **Friday, 20.10.2017, 13:30** (English).

1. Simultaneous eigenstates

4+2+2 Punkte

- (a) Let \hat{A} and \hat{B} be two operators which commute. Show that there exists a common set of eigenstates of the two operators. Distinguish between the case where the states are non-degenerate and n-fold degenerate.
- (b) Assume now that $|\Psi\rangle$ is a simultaneous eigenstate of \hat{A} and \hat{B} , and that \hat{A} and \hat{B} anti-commute: $\{\hat{A}, \hat{B}\} \equiv \hat{A}\hat{B} + \hat{B}\hat{A} = 0$. What can you say about the eigenvalues of the two operators?
- (c) Give a concrete example of your result from part (b) using the parity and momentum operators.

2. Symmetric double-well potential

2+4 Punkte

Consider a symmetric rectangular double-well potential

$$V(x) = \begin{cases} \infty, & |x| > a + b, \\ 0, & a < |x| < a + b, \\ V_0, & |x| < a, \end{cases}$$

with $V_0 > 0$.

- (a) Write down the solution to the Schrödinger equation in the different regions and use suitable boundary conditions to construct a solution valid for all x .
Hint: Due to the symmetry of the problem you can choose solutions which are also eigenstates of parity.
- (b) Assuming V_0 is very large, calculate the energies of the ground state and the first excited state.

3. Supersymmetry

2+2+2 Punkte

In particle physics, supersymmetry is a hypothetical relationship between the fermionic and bosonic degrees of freedom, which could resolve some of the most important problems of the Standard Model, such as the hierarchy problem. More generally, supersymmetry can be formulated with the help of a supercharge operator. The Hamiltonian \hat{H} is supersymmetric if there exists a supercharge operator \hat{Q} so that $\hat{H} = \{\hat{Q}, \hat{Q}^\dagger\}$ and $\{\hat{Q}, \hat{Q}\} = \{\hat{Q}^\dagger, \hat{Q}^\dagger\} = 0$.

- (a) Show that the supercharge is conserved i.e. $[\hat{H}, \hat{Q}] = [\hat{H}, \hat{Q}^\dagger] = 0$.
- (b) Assume that $|\psi\rangle$ is an eigenstate of \hat{H} with eigenenergy $E > 0$. Notice that because $-i(\hat{Q}^\dagger - \hat{Q})$ commutes with \hat{H} , $|\psi\rangle$ can be chosen to be a simultaneous eigenstate of $-i(\hat{Q}^\dagger - \hat{Q})$. Show that there exists a superpartner eigenstate $|\psi_2\rangle = (\hat{Q} + \hat{Q}^\dagger)|\psi\rangle$ with the same eigenenergy E . Show that $|\psi_2\rangle$ corresponds to a different eigenvalue of the operator $-i(\hat{Q}^\dagger - \hat{Q})$ than $|\psi\rangle$. Therefore, all $E > 0$ eigenstates are doubly degenerate. Explain why the $E = 0$ eigenstate can be nondegenerate.
- (c) Consider a Hamiltonian

$$\hat{H} = \begin{pmatrix} \hat{H}_- & 0 \\ 0 & \hat{H}_+ \end{pmatrix} = \begin{pmatrix} \frac{\hat{p}^2}{2m} + W^2(x) - W'(x)\frac{\hbar}{\sqrt{2m}} & 0 \\ 0 & \frac{\hat{p}^2}{2m} + W^2(x) + W'(x)\frac{\hbar}{\sqrt{2m}} \end{pmatrix}.$$

Show that \hat{H} is supersymmetric and the corresponding supercharge operator is

$$\hat{Q} = \left(i\frac{\hat{p}}{\sqrt{2m}} + W(x) \right) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Assume that $|\psi_-\rangle = \psi_-(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenstate of \hat{H} with eigenenergy $E > 0$, so that $\psi_-(x)$ is an eigenfunction of \hat{H}_- i.e. $\hat{H}_-\psi_-(x) = E\psi_-(x)$. Show that its superpartner eigenstate $|\psi_+\rangle \equiv (\hat{Q} + \hat{Q}^\dagger)|\psi_-\rangle = \psi_+(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ where $\hat{H}_+\psi_+(x) = E\psi_+(x)$.

4. Bonus Problem

+3 Extra Punkte

Find the eigenenergies of the bound states of the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{\hbar^2}{2mx_0^2} \left(a^2 - \frac{a(a+1)}{\cosh^2(x/x_0)} \right),$$

where a is a dimensionless constant with $a > 1$. How many bound states are there? What happens if $a = 1$?

Hint: Consider the previous problem with $W_a(x) = a\frac{\hbar}{\sqrt{2m}x_0} \tanh(x/x_0)$. Notice that if $a > 0$, the Hamiltonian \hat{H}_- has a zero-energy solution of the form

$$\psi_-(x) = C \exp \left(-\frac{\sqrt{2m}}{\hbar} \int_0^x W_a(x') dx' \right).$$