
Quantum Field Theory of Many-Particle Systems - Problem Set 1

Summer Semester 2024

Due: The problem set will be discussed in the first tutorial.

Internet: The problem sets can be downloaded from
https://home.uni-leipzig.de/stp/QFT_of_MPS_SS24.html

1. Field operators

5 Punkte

The operators a_k^\dagger and a_k create or annihilate single particle states with momentum k , respectively. They obey the commutation relations $[a_k, a_{k'}]_\zeta = 0$, and $[a_k, a_{k'}^\dagger]_\zeta = \delta_{k,k'}$ with $\zeta = 1$ for bosons and $\zeta = -1$ for fermions. The field operator $\Psi(x)$ is defined as the Fourier transform

$$\Psi(x) = \frac{1}{\sqrt{L}} \sum_k a_k e^{ikx}.$$

Show that $\Psi(x)$ and its Hermitian conjugate $\Psi^\dagger(x)$ obey the commutation relations

$$[\Psi(x), \Psi^\dagger(y)]_\zeta = \delta(x - y).$$

2. Second Quantisation with field operators

10 Punkte

A many-particle state is described by the Hamiltonian

$$H = \sum_{j=1}^N \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_j^2} + V(x_j) \right) + \sum_{i < j} V(x_i - x_j),$$

and the wave function $\varphi_\alpha(x_1, x_2, \dots, x_N)$ is an eigenstate of H with eigenvalue E_α . In second quantised notation, the wave function φ_α is described by a state vector

$$|\varphi_\alpha\rangle = \int dx_1 dx_2 \dots dx_N \varphi_\alpha(x_1, x_2, \dots, x_N) \Psi^\dagger(x_1) \Psi^\dagger(x_2) \dots \Psi^\dagger(x_N) |0\rangle,$$

with commutation relations of Ψ and Ψ^\dagger as in problem 1, supplemented by the condition that the operator $\Psi(x)$ annihilates the vacuum state, $\Psi(x)|0\rangle = 0$. In second quantisation, the Hamiltonian reads

$$H_s = \int dx \Psi^\dagger(x) \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right) \Psi(x) + \frac{1}{2} \int dx dy \Psi^\dagger(x) \Psi^\dagger(y) V(x - y) \Psi(y) \Psi(x).$$

Show that

$$H_s |\varphi_\alpha\rangle = E_\alpha |\varphi_\alpha\rangle.$$