Bulk-edge coupling in the non-abelian $\nu = 5/2$ quantum Hall interferometer

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Recently schemes for probing non-abelian statistics in the quantum Hall effect are based on geometries where current-carrying quasiparticles flow along edges that encircle bulk quasiparticles, which are localized. Here we consider one such scheme, the Fabry-Perot interferometer, and analyze how its interference patterns are affected by a coupling that allows tunneling of neutral Majorana fermions between the bulk and edge. While at weak coupling this tunneling degrades the interference signal, we find that at strong coupling, the bulk quasiparticle becomes essentially absorbed by the edge and the interference signal is fully restored. Furthermore, we find that the strength of the coupling can be tuned by the source-drain voltage.

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Recently, interference experiments were proposed as a way to examine the non-abelian nature of quasiparticles in the $\nu = 5/2$ quantum Hall state[1–4]. The most dramatic signature of non-abelian statistics is expected to be seen in the interference of back-scattering amplitudes from two constrictions in a long Hall bar. (See inset in Fig. 1.) The two constrictions enclose a “cell”, whose area may be varied by means of a side-gate. The bulk is assumed to host a number $N_{qp}$ of localized quasiparticles, that do not take part in electronic transport, and have no tunnel coupling to the edge. In the limit of weak back-scattering, when $N_{qp}$ is even the two back-scattering amplitudes interfere coherently, while when $N_{qp}$ is odd they are incoherent, and thus do not interfere. In the former case, the back-scattered current oscillates with the area of the cell, while in the latter case it does not. This difference reflects the non-abelian nature of the quasiparticles.

In a real system, some degree of coupling between the edge and quasiparticles localized in the bulk is unavoidable. One may suspect that such a coupling would blur the distinction between bulk and edge quasiparticles, obscure the very definition of $N_{qp}$, and endanger the possibility of observing the even-odd effect. Furthermore, it may destabilize the topological qubits which are proposed to be realized in such a geometry. [2] In this work, we find that for strong enough coupling the even-odd effect survives, and that the dimensionless parameter which determines the coupling strength is externally controllable; it increases with decreasing voltage.

Since both the 5/2 edge and the quasiparticles consist of both neutral Majorana fermionic and charged bosonic degrees of freedom, several types of edge to bulk coupling are possible. We expect that at low energies, tunneling that involves a charge will generally be suppressed due to the Coulomb energy. Thus in this work we will focus on tunneling of the neutral Majorana mode from the bulk to the edge, and on the resulting effect on the interference.

The system we consider [3–6] is a Hall bar lying parallel to the $x$-axis (See Fig. 1). Two constrictions are located at $x = -b$ and $x = b$. We focus on a simple case where there are two quasiparticles, $N_{qp} = 2$, localized at $x = 0$, between the two constrictions, with one of the quasiparticles coupled to the upper edge and the other coupled to the lower edge. Ref. [7] considers the case of $N_{qp} = 1$ in the weak tunneling limit.

When the two localized quasiparticles are decoupled from the edge they form a two level system, and the ground state is doubly degenerate. The interference patterns that are seen in the two orthogonal ground states are mutually shifted by a phase $\pi$. Then, at temperature $T = 0$ the magnitude of the interference term depends on the ratio of two time scales, one determined by the voltage $t_V = \hbar/e^*V$, and the other being the time associated with motion between the two constrictions $t_b = 2\hbar/v$, where $v$ is a characteristic edge mode velocity. When analyzing the effect of bulk-edge coupling we will focus on the case of low voltage, $t_V \gg t_b$, where the interference is most clearly seen.

In the absence of edge-bulk coupling the system cannot switch from one ground state to another. Thus if it is prepared in one ground state, repetitive measurements of the interference would show the same interference pattern. However, when the interference term is averaged over the two possible ground states, e.g., by measuring the interference with a random choice of the initial ground state, the average is zero. When the coupling of the bulk two-level system to the edge is turned on, the average value of the interference term becomes non-zero, and the correlation function between consecutive measurements is strongly modified. Denoting the coupling strengths between the localized Majorana particles and their respective edges by $\lambda_u$ and $\lambda_d$, [defined precisely in Eq. (1) below], we obtain corresponding time scales $t_{\lambda u(d)} = (\pi v_m^2/2\nu_m)^{-1}$, where $v_m$ is the velocity of the Majorana modes on the edges. In the limit of weak coupling, where $t_{\lambda u(d)} \gg t_V$, we may use a perturbation analysis, and we find that the average value of the interference is proportional to $(t_V)^{1/2}(t_V/t_{\lambda}) \log^2[t_{\lambda}/t_V]$, where we have assumed that $t_{\lambda u}$ and $t_{\lambda d}$ are comparable in magnitude, and $t_{\lambda}$ is their geometric mean. As the coupling is increased, or as the voltage is lowered, the perturbative analysis breaks down. We then carry out a numerical analysis, which suggests that in the limit $t_V/t_{\lambda} \rightarrow \infty$ the full magnitude of the interference term...
is retrieved. In effect, the two bulk quasiparticles become then a part of the edge, and $N_{I,F}$ reduces from two to zero. (We find a similar effect for a single quasiparticle strongly coupled to an edge.) In contrast to the build-up of the average interference term as the coupling gets stronger, the fluctuating part of the interference pattern is weakened by the coupling, and its characteristic correlation time becomes $t_\lambda$, which decreases with increased coupling.

For the derivation of these results, we introduce the relevant Hamiltonians, derive the proper quasiparticle tunneling operator, perturbatively analyze the weak coupling limit, and finally numerically analyze the strong coupling limit. To describe the effect of the bulk quasiparticles, we represent the tunneling operator both in a local form using the $\sigma$ operator of the Ising Conformal Field Theory (CFT) [9], and in an alternative non-local form using a “parity operator” that measures the parity of the number of electrons encircled by the interfering paths of a back-scattered quasiparticle.

The Hamiltonian density for a chiral Luttinger liquid with velocity $v_0$, both at $\nu = 1$ and $\nu = 0$, is described by the Hamiltonian density $H_{\text{LL}} = \frac{1}{4\pi} \psi'(x)\psi'(x) - (\partial_x)\psi'(x) + \frac{e}{\hbar} \sigma(x)\delta(x)$. For simplicity we set the velocities of the Majorana modes to be equal and opposite $u_m = -v_m$. Furthermore, we set $v_m = 1$ when no confusion results. Each of the two localized bulk quasiparticles carries a zero quasi-particle charge. Correspondingly, the current operator is $J = i e^\psi d(\mp b) - e^\psi d(\mp b))/\sqrt{\hbar}$ are the charge part of the tunneling operator, operating on the charge mode. The Aharonov-Bohm phase is absorbed into the relative phase between the tunneling coefficients $\eta_{L,R}$. The neutral parts of the tunneling operators are $N_L = \sigma_u(-b)\sigma_d(-b)$ and $N_R = \sigma_u(b)\sigma_d(b)$. For the present purpose, the $\sigma$ operators are defined through their operation on the Majorana fermion fields as [8]

$$\psi_r(y)\sigma_r(x) = -\text{sgn}(x_0 - y)\psi_r(x)\sigma_r(y)$$

with $r = u, d$. The factor of $\gamma_u\gamma_d$ in the second term of Eq. (2) is included to account for the wrapping of a tunneling quasiparticle at position $x = b$ around the two localized quasiparticles; it introduces the $\pi$ phase shift between the interference patterns corresponding to the two eigenvectors of $\gamma_u\gamma_d$.

The neutral mode part of the tunneling operators may also be expressed in a non-local form through the Majorana fermions along the two edges in a way which we find to be both illuminating and useful. This approach is based on the description of the $\nu = 5/2$ state as a $p$-wave superconductor of composite fermions [1]. Within this description the bulk is a superconductor, with the localized quasiparticles being vortices in that superconductor. A tunneling of a quasiparticle from one edge to another at position $x_0$ involves a tunneling of a vortex, and that introduces a twist into the phase of the order parameter: for all points in the region $x < x_0$, the phase is shifted by $2\pi$, while for all points in the region $x > x_0$ the phase is unaffected by the vortex motion (up to an unimportant global gauge redefinition). To implement this shift of the phase, we recognize that the field is canonically conjugate to the Cooper-pair density field, which at zero temperature is just half the electron density field. The operator that implements the required shift in the phase is then

$$P(\nu, x_0) = e^{i\int_{x_0} dx \psi(x)}$$

Since the operator $\int_{x_0} dx \psi(x)$ has only integer eigenvalues, the operator $P(\nu, x_0)$ is nothing but a Parity Operator which measures the parity of the number of electrons to the left of $x_0$. Eq. (2) can thus be rewritten as $T = e^{i e^\psi V^i}[\eta_L^c C_L P(-\infty, -b) + \eta_R^c C_R P(-\infty, b)]$ as we shall see below.

Since the bulk of the system is gapped, and since all particles in the superconducting ground state are paired, the parity operator only has contributions from localized neutral modes and from the neutral mode along the edge. The operator $\gamma_{u,d}$ in the second term of Eq. (2) precisely counts the parity of the number of fermions in the localized bulk quasiparticles to the left of $x = b$. Counting the fermions along the edge is a bit more complicated but is achieved by constructing a complex Fermi field $\psi(x) = \psi_u(x) - i\psi_d(x)$ and $\psi^*(x) = \psi_u(x) + i\psi_d(x)$, such that the edge contribution to the parity operator is

$$P_{\text{edge}}(-\infty, x_0) = e^{i\pi \int_{x_0} dx \psi^*(x)} \psi(x)$$

It is easy to see that Eq. (3) holds when the operators $\sigma_r(x_0)$ are replaced by $P_{\text{edge}}$. The eigenvalues of the latter are
\( \pm 1 \), since the eigenvalues of \( \int_{x_0} x \, d\psi_0^\dagger(x)\psi_0(x) \) are integers. The application of either \( \psi_d(y) \) or \( \psi_u(y) \) on an eigenstate of \( \int_{x_0} x \, d\psi_0^\dagger(x)\psi_0(x) \) changes the eigenvalue by \( \pm \theta(x_0 - y) \), and hence (3). Altogether, then, we have \( N_{L(R)} = P_{\text{edge}}(-\infty, +b) \).

To calculate the current-voltage characteristics in the weak back-scattering limit, we use standard [6] perturbation theory in the tunneling strength to yield \( I = \frac{1}{4h} \int_{-\infty}^{\infty} dt \langle [J(0), H_{\text{int}}(t)] \rangle \). With some algebra, the interference term that results is

\[
I_{\text{int}} = \frac{2e^2 \eta R \eta_0}{\hbar^2} \int_{-\infty}^{\infty} dt \, e^{-i e^* V t / \hbar} \times \langle [C_L^+(t) N_L(t), C_R(t) N_R(t) \gamma_u(0) \gamma_d(0)] \rangle .
\]  

(6)

For \( -e^* V > 0 \), only the first term of the commutator, with \( t \)-dependent operators to the left, will contribute to the integral. The correlator of the charged operators (the \( C \)'s) and that of the neutral operators (the \( N \)'s and \( \gamma \)'s) factorize. The correlator of the charged operator \( \langle C_L^+(t) \rangle \langle C_R(0) \rangle \) is just the parity-parity correlator \( \langle P(-\infty, -b; t) P(-\infty, b; t = 0) \rangle \). In the absence of edge-bulk coupling, this correlator breaks into a product \( \langle N_L(t) N_R(0) \rangle \gamma_u(0) \gamma_d(0) \rangle \). We define \( I_{\text{N}0}(t) = \langle N_L(t) N_R(0) \rangle \gamma_u(0) \gamma_d(0) \rangle \). We note that the reduction factor \( I_{\text{R}}(t) = I_{\text{N}0}(t) I_{\text{N}0}^{-1}(t) \).

We now turn to analyze the reduction factor in various regimes of bulk-edge coupling. The two edge theories \( (u, d) \) factorize with the exception of a constraint demanding conservation of overall parity, which corresponds to choosing the same fusion channel for the two edges (see Ref. [8]); if there was only one bulk impurity in the cell, the fusion channels would be different on the two edges), and we can write the correlator \( I_{\text{N}}(t) = I_u(t) I_d(t) \). In the limit of weak coupling, we may use perturbation theory. We expand the time evolution operator to lowest order in \( \lambda \). The perturbed correlators can be written as correlators in an unperturbed theory

\[
I_u(t) = \lambda \int dt' \langle T \sigma_u(-b, t) \tau_u(b, 0) \psi_u(0, t') \rangle \langle T \gamma_u(0) \gamma_u(t') \rangle.
\]  

(7)

where the time integration contour starts at \( -\infty \) goes up to \( t \) across the real axis then back to \( -\infty \) and \( T \) represents the appropriate (Keldysh) time ordering of operators.

The correlators of the type \( \langle \sigma \psi \rangle \) are well known from conformal field theory [9]: \( \langle \sigma(z_1) \sigma(z_2) \psi(z_3) \rangle = \frac{1}{\sqrt{2}} (z_{12})^{3/8} (z_{23} z_{13})^{1/2} \) where \( z_{ij} \equiv z_i - z_j \) and \( z = x + i \tau \) in imaginary time, which then needs to be continued back to real time. Substituting this correlator in Eq. (7) and noting that at the unperturbed level \( \langle \gamma_u(0) \gamma_u(t) \rangle = 1 \) we find the time integral to be logarithmically divergent. However, when the correlator \( \langle \gamma_u(0) \gamma_u(t) \rangle \) is itself calculated in perturbation theory, it is found to decay at a time scale of order \( t_b \) and thus provides a natural cutoff for the time integration. Evaluating the integrals with the cutoff yields (in the limit of small \( t_b \)) the leading contribution of the upper edge to the parity correlator

\[
I_u(t) = (\delta + i t)^{3/8} \left\{ \lambda_u \sqrt{2} \left[ -i \log(|t|/t_{\lambda}) - \pi \text{sgn} t \right] \right\},
\]  

(8)

independent of the details of the cutoff. When we consider coupling of impurities to both edges, we obtain a similar expression for \( I_d \). The reduction factor defined above is then

\[
R(t) = 2\lambda_u \lambda_d t \log(|t|/t_{\lambda}) \log(|t|/t_{\lambda}) + \ldots
\]  

(9)

Including the contributions from the charge modes and \( T^0_\lambda \) in Eq. (6), results in an interference current proportional to \( \lambda_u \lambda_d V^{-3/2} \log(t_{\lambda}) \log(t_{\lambda}) \). We note that the reduction factor Eq. (9) is a product of separate contributions from the upper and lower edge, respectively. Using one of these factors (say, for the upper edge) to calculate the interference current, the corresponding logarithmic factor disappears, and in the weak coupling limit, the interference current is proportional to \( \lambda V^{-1} \), in agreement with the result in Ref. [7].

In order to analyze the strong coupling limit with either \( t_b \) or \( t_v \) of the order of \( t_{\lambda} \), we numerically study a lattice version of our model. We start with a tight-binding Hamiltonian for one-dimensional complex fermions without the localized modes: \( H = -v_{\lambda} a^{-1} \sum_j (c_j^+ c_j + h.c.) \). Here, \( a \) is the lattice constant, and the operators obey the usual anti-commutation relations \( [c_j^+, c_j] = \delta_{ij} \). We study the model at half filling with a Fermi wave vector \( k_F = \pi/2 \). The fermions created by \( c_j \) can be decomposed into two Majorana species \( \gamma_j = e^{i \pi/2} c_j + h.c., \), which we define continuum fermions, using \( \gamma_j \) alone, via

\[
\psi_{u,d}(ja) = \frac{1}{\sqrt{a}} \sum_j (\pm 1)^j f(j - j') \gamma_{j'},
\]  

(10)

where \( f(j) \) is a Gaussian with a width large compared to the lattice spacing and subject to the normalization \( \sum_j f(j) = \sqrt{a} \). Using this mapping one can now include the coupling to the localized modes as in Eq. (1).

The parity operator for a set of lattice sites \( \{j\} \) can be written as the product over sites of operators \( 2 e^{i \pi/2} c_j - 1 = i \gamma_j \tilde{\gamma}_j \). The parity operator for the localized modes has a similar form. The Hamiltonian is a sum \( H = H_1 + H_2 \), and one finds \( \{\gamma_j, \tilde{\gamma}_j\} = 0 \), hence the \( \gamma \) and \( \tilde{\gamma} \) completely decouple from each other. As the edge theory of the \( \nu = \frac{5}{2} \) quantum Hall state contains only one Majorana degree of freedom, and as the localized modes couple to \( \gamma \) fermions only, we can completely discard the \( \tilde{\gamma} \) degrees of freedom. In this way, the lattice form for the edge parity for a region \( [x_1, x_2] \) becomes

\[
P_{\text{edge}}(x_1, x_2) = \prod_{x_1 \leq x_j \leq x_2} \sqrt{t} \gamma_j .
\]  

(11)
The numerical results presented in Fig. 1 were obtained using this numerical technique deviate less than 0.3% from each other, as shown in Fig. 1. In the regime of weak coupling (not shown in Fig. 1), we find excellent agreement between our numerical results and the perturbative solution Eq. (9).

Fig. 1 displays the imaginary time reduction factor $\tilde{R}(\tau)$ for intermediate and strong coupling, for an interferometer size $b = t_\lambda/4$. $\tilde{R}(\tau)$ is related to the real time reduction factor via the analytic continuation $R(t) = \tilde{R}(\tau \rightarrow it + \delta)$. The reduction factor $\tilde{R}(\tau)$ monotonically increases with increasing time. At $\tau \approx t_\lambda$, there is a crossover from parity reduction determined by the interferometer size $b = t_\lambda/4$ to parity reduction determined by the time. At large times $\tilde{R}(\tau)$ seems to saturate at a value of one, implying that its analytic continuation $R(t)$ saturates near one as well. Note that when $R(t) = 1$ the visibility of the interference is the same as it would have been in the absence of the two bulk quasiparticles. Similar results are expected if we have one strongly coupled localized mode inside the interferometer path, and a second localized mode, of arbitrary coupling, outside the interferometer. We attribute the re-emergence of the interference as the bulk-edge coupling gets strong to the correlations that develop between the occupation of the fermionic mode associated with the two quasi-particles and the occupation of the region of the edge at a distance $v_m t_\lambda$ from the coupling point. Each of these occupations strongly fluctuates due to the coupling, but their fluctuations are strongly correlated.

In conclusion, we found that when the coupling between the Majorana mode associated with a localized $e^\star = e/4$ charged quasiparticle and an adjacent edge is strong, so that the characteristic time $t_\lambda$ is short compared to the scale $t_V = \hbar e^\star / V$ set by the voltage, it appears as if the localized quasiparticle has become part of the edge. For an interference path enclosing the quasiparticles, the interference visibility should have essentially the same strength as if the quasiparticle were not there. For weak coupling, the time-averaged interference intensity is reduced by a factor $\propto (t_V/t_\lambda) \log^2(t_\lambda/t_V)$ in the case of two localized quasiparticles inside the loop, coupled respectively to the two edges with similar strength. In this way, the tunability of the dimensionless parameter $t_V/t_\lambda$ becomes a useful tool for the experimental study of interference effects in non-abelian Fabry-Perot devices.

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