Climate Dynamics (Summer Semester 2018)
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Today's Lecture (Lecture 6): Lorenz cycle, entropy cycle

Reference
Peixoto and Oort, Ch. 14, 15
Lorenz, E.N. (1955), *Available Potential Energy and the Maintenance of the General Circulation*. Tellus, **7**: 157–167. doi:10.1111/j.2153-3490.1955.tb01148.x

Lorenz cycle

The atmosphere as a heat engine

The radiative energy imbalance leads to atmospheric motion (kinetic energy). We can think of the atmosphere as a heat engine that is supplied with energy in the form of radiative, sensible, and latent heat fluxes and converts that energy into motion (kinetic energy) in its mean and eddy circulations. The kinetic energy is eventually dissipated by friction.
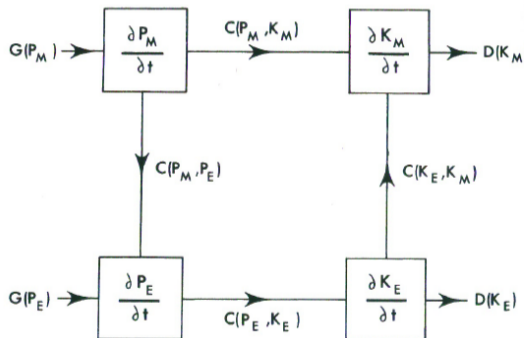
$$\frac{\partial P_M}{\partial t} = G(P_M) - C(P_M, P_E) - C(P_M, K_M) \qquad (2.102)$$

$$\frac{\partial P_E}{\partial t} = G(P_E) + C(P_M, P_E) - C(P_E, K_E) \qquad (2.103)$$

$$\frac{\partial K_M}{\partial t} = C(P_M, K_M) + C(K_E, K_M) - D(K_M) \qquad (2.104)$$

$$\frac{\partial K_E}{\partial t} = C(P_E, K_E) - C(K_E, K_M) - D(K_E) \qquad (2.105)$$

where

$G(i)$ is generation of potential energy by heat fluxes

$C(i, j)$ is conversion between energy forms

$D(i)$ is dissipation of kinetic energy

The following slides summarize the derivation of the generation, conversion, and dissipation terms
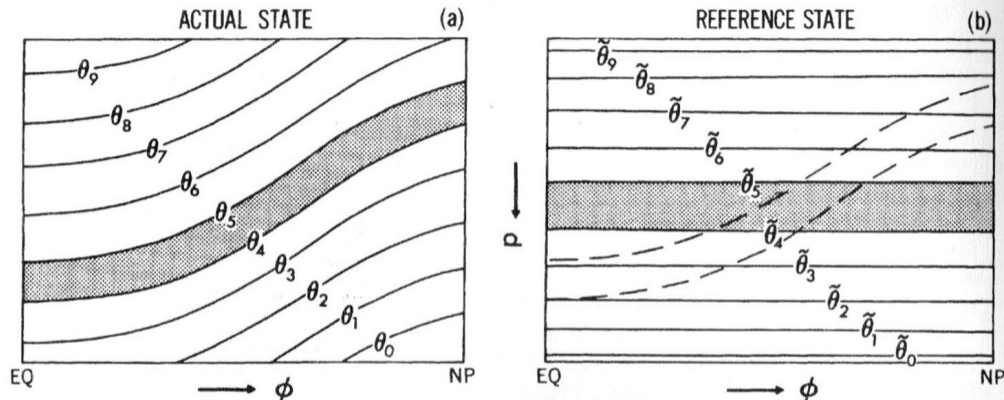
Potential energy

Total potential energy

Sum of the internal (thermal) and geopotential energy is called the potential energy:

$$
\begin{aligned}
\text{potential energy} &= \int \rho c_v T \, dx \, dy \, dz + \int \rho g z \, dx \, dy \, dz \\
&= c_p \int \rho T \, dx \, dy \, dz \\
&= \frac{c_p}{g} \int T \, dx \, dy \, dp \\
&= \frac{c_p}{g p_0^\kappa (\kappa + 1)} \int p^{\kappa+1} \, dx \, dy \, d\theta
\end{aligned}
\tag{2.106}
$$

Availability

Some of the potential energy is available for conversion to kinetic energy, but much of it is not – think of a stably stratified barotropic atmosphere.

Available potential energy

FIGURE 14.2. Schematic diagram of the actual state (a) and the reference state (b) used to compute the available potential energy in the atmosphere. The reference state is obtained by an isentropic redistribution of the atmospheric mass so that the isentropic surfaces become horizontal. During the redistribution the area (i.e., the mass) between two isentropic surfaces remains the same. Note that in a stable state the potential temperature θ increases with height.

Available potential energy

Minimum energy state of the atmosphere

The available potential energy is the potential energy relative to the lowest energy state that the atmosphere can achieve using only adiabatic processes. In this reference state, the isentropes are parallel to the isobars.

Potential temperature coordinates

Use potential temperature as a vertical coordinate (in the mean state of the atmosphere, θ increases monotonically with height). $p(x, y, \theta)$ is the weight of the atmosphere column above the isentropic surface θ, assuming hydrostatic equilibrium.

Reference-state pressure

Global average pressure $\tilde{p}(\theta)$ over a θ surface:

$$\tilde{p}(\theta) = \frac{\int p(x, y, \theta) \, dx \, dy}{\int dx \, dy} \tag{2.107}$$

In the reference state, $p(x, y, \theta) = \tilde{p}(\theta)$; denoting the reference state with a subscript r,

$$p_r(x, y, \theta) = \tilde{p}(\theta) \tag{2.108}$$

Available potential energy

Subtracting the potential energy in the reference state from the actual state using (2.106), we find the *available potential energy*:

$$P = \frac{c_p}{g(1+\kappa)p_0^{\kappa}} \int \left(p^{\kappa+1} - \tilde{p}^{\kappa+1} \right) \, dx \, dy \, d\theta \qquad (2.109)$$

Writing $p(x, y, \theta) = \tilde{p}(x, y, \theta) + \hat{p}(x, y, \theta)$ as the sum of global mean and deviation from global mean, we can expand the integrand in (2.109) as

$$p^{\kappa+1} - \tilde{p}^{\kappa+1} = \tilde{p}^{\kappa+1} \left\{ \left(\frac{p}{\tilde{p}} \right)^{\kappa+1} - 1 \right\} = \tilde{p}^{\kappa+1} \left\{ \left(1 + \frac{\hat{p}}{\tilde{p}} \right)^{\kappa+1} - 1 \right\}$$

$$\approx \tilde{p}^{\kappa+1} \left\{ (\kappa+1) \frac{\hat{p}}{\tilde{p}} + \frac{\kappa}{2} (\kappa+1) \left(\frac{\hat{p}}{\tilde{p}} \right)^2 + \dots \right\} \qquad (2.110)$$

for $\hat{p} \ll \tilde{p}$. The leading term vanishes in the global integral, yielding

$$P \approx \frac{c_p}{g(1+\kappa)p_0^{\kappa}} \int \tilde{p}^{\kappa+1} \frac{\kappa}{2} (\kappa+1) \left(\frac{\hat{p}}{\tilde{p}} \right)^2 \, d\theta \, dx \, dy = \frac{1}{2} \frac{c_p \kappa}{g p_0^{\kappa}} \int \tilde{p}^{\kappa+1} \left(\frac{\hat{p}}{\tilde{p}} \right)^2 \, d\theta \, dx \, dy \qquad (2.111)$$

Available potential energy

Transforming (2.111) into the p coordinate system using

$$\hat{\theta} = -\frac{\partial \tilde{\theta}}{\partial p}\hat{p} \qquad \text{and} \qquad (2.112)$$

$$d\theta = -\frac{\partial \theta}{\partial p}\, dp \qquad (2.113)$$

we find

$$P = \frac{1}{2}\frac{c_p \kappa}{g p_0^\kappa} \int \tilde{p}^{\kappa+1}\frac{\hat{\theta}^2}{\tilde{p}^2}\left(\frac{\partial \tilde{\theta}}{\partial p}\right)^{-2}\left(-\frac{\partial \theta}{\partial p}\right)\, dp\, dx\, dy \approx -\frac{1}{2}\frac{c_p \kappa}{g p_0^\kappa}\int \tilde{p}^{\kappa-1}\hat{\theta}^2\left(\frac{\partial \tilde{\theta}}{\partial p}\right)^{-1}\, dp\, dx\, dy$$

$$= -\frac{1}{2}\frac{R_d}{g}\int \frac{\tilde{T}}{\tilde{\theta}\tilde{p}}\left(\hat{\theta}\right)^2\left(\frac{\partial \tilde{\theta}}{\partial p}\right)^{-1}\, dp\, dx\, dy \quad \text{(using } \kappa = R_d/c_p \text{ and } p^\kappa/p_0^\kappa = T/\theta\text{)} \qquad (2.114)$$

Expressing the atmospheric stability $\partial \tilde{\theta}/\partial p$ in terms of temperature lapse rates $\tilde{\Gamma} = -\partial \tilde{T}/\partial p$ and $\Gamma_d = g/c_p$, we find that the available potential energy depends on the deviations of temperature from the global mean:

$$P = \frac{1}{2}\left(\Gamma_d - \tilde{\Gamma}\right)^{-1}\int \frac{\tilde{T}}{\tilde{\theta}^2}\left(\theta - \tilde{\theta}\right)^2\, dp\, dx\, dy \approx \frac{1}{2}\left(\Gamma_d - \tilde{\Gamma}\right)^{-1}\frac{1}{\tilde{T}}\int \hat{T}^2\, dp\, dx\, dy$$

$$= \frac{c_p}{2}\Gamma_d\left(\Gamma_d - \tilde{\Gamma}\right)^{-1}\frac{1}{\tilde{T}}\int \hat{T}^2\, \rho\, dx\, dy\, dz = \frac{c_p}{2}\Gamma_d\left(\Gamma_d - \tilde{\Gamma}\right)^{-1}\frac{1}{\tilde{T}}\int \hat{T}^2\, dm \qquad (2.115)$$

Available potential energy

Taking the zonal and temporal mean of (2.115) using the decomposition

$$\hat{T}^2 = \left(T - \tilde{T}\right)^2 = T^2 - 2T\tilde{T} + \tilde{T}^2 \tag{2.116}$$

$$\left[\overline{(T - \tilde{T})^2}\right] = \left(\left[\overline{T}\right] - \overline{\tilde{T}}\right)^2 + \left[\left(\overline{T^*}\right)^2\right] + \left[\overline{T'^2}\right] \tag{2.117}$$

we find

$$P = P_M + P_E = P_M + P_{TE} + P_{SE} = \frac{c_p}{2} \int \gamma \left(\left[\overline{T}\right] - \overline{\tilde{T}}\right)^2 \, dm + \frac{c_p}{2} \int \gamma \left[\overline{T'^2}\right] \, dm + \frac{c_p}{2} \int \gamma \left[\overline{(T^*)^2}\right] \, dm \tag{2.118}$$

with the stability parameter $\gamma = \Gamma_d \tilde{T}^{-1} (\Gamma_d - \tilde{\Gamma})^{-1}$; the decomposition is into *mean*, *transient-eddy*, and *stationary-eddy* contributions to the available potential energy.

Kinetic energy balance equations

Consider a field A which has a source or sink term S. The balance equation for A is:

$$\frac{\partial A}{\partial t} = -\vec{V} \cdot \nabla A - \omega \frac{\partial A}{\partial p} + S \tag{2.119}$$

The kinetic energy terms involve $1/2([u]^2 + [v]^2)$ for the mean kinetic energy and $1/2([u^{*2}] + [v^{*2}])$ for the eddy kinetic energy. To get a balance equation for $[A]^2$, we multiply (2.119) by $[A]$:

$$[A]\frac{\partial A}{\partial t} = -\vec{V}[A] \cdot \nabla A - \omega[A]\frac{\partial A}{\partial p} + [A]S$$

$$= -\vec{V}[A] \cdot \nabla[A] - \omega[A]\frac{\partial[A]}{\partial p} + [A]S - [A]\nabla \cdot \left(\vec{V}A^*\right) - [A]\frac{\partial}{\partial p}\left(\omega A^*\right) + [A]A^*\nabla \cdot \vec{V} + [A]A^*\frac{\partial \omega}{\partial p}; \tag{2.120}$$

note that the second line uses the decomposition (2.91). Taking the zonal mean of (2.120), we find

$$\frac{\partial}{\partial t}\left(\frac{1}{2}[A]^2\right) = -\left[\vec{V} \cdot \nabla\left(\frac{1}{2}[A]^2\right)\right] - [\omega]\frac{\partial}{\partial p}\left(\frac{1}{2}[A]^2\right) + [A][S] - [A]\left[\nabla \cdot \left(\vec{V}^*A^*\right)\right] - [A]\frac{\partial}{\partial p}[\omega^*A^*]$$

$$= -\nabla \cdot \left(\left[\vec{V}\right]\frac{1}{2}[A]^2\right) - \frac{\partial}{\partial p}\left([\omega]\frac{1}{2}[A]^2\right) + [A][S] - \nabla \cdot \left([A]\left[\vec{V}^*A^*\right]\right) - \frac{\partial}{\partial p}([A][\omega^*A^*])$$

$$+ \frac{1}{2}[A]^2 \nabla \cdot \left[\vec{V}\right] + \frac{1}{2}[A]^2\frac{\partial}{\partial p}[\omega] \quad \{= 0 \text{ by continuity equation}\}$$

$$+ \left[\vec{V}^*A^*\right]\nabla[A] + [\omega^*A^*]\frac{\partial[A]}{\partial p} \tag{2.121}$$

Kinetic energy balance equation – mean field

We can simplify further using

$$\nabla \cdot \left(\left[\vec{V} \right] [A]^2 \right) = \frac{\partial}{R_E \partial \phi} \left([v][A]^2 \right) \tag{2.122}$$

to obtain the balance equation for $[A]^2$:

$$
\begin{aligned}
\frac{\partial}{\partial t} \left(\frac{1}{2} [A]^2 \right) = & -\frac{\partial}{R_E \partial \phi} \left([v] \frac{1}{2} [A]^2 \right) - \frac{\partial}{\partial p} \left([\omega] \frac{1}{2} [A]^2 \right) && \text{flux divergence terms} \\
& + [A] [S] && \text{sources/sinks} \\
& - \frac{\partial}{R_E \partial \phi} \left([A] [v^* A^*] \right) - \frac{\partial}{\partial p} \left([A] [\omega^* A^*] \right) && \text{flux divergence terms} \\
& + [v^* A^*] \frac{\partial [A]}{R_E \partial \phi} + [\omega^* A^*] \frac{\partial [A]}{\partial p} && \text{conversion terms} \qquad (2.123)
\end{aligned}
$$

Note that in integrals over the entire atmosphere, the flux divergence terms vanish (because the ϕ boundaries are periodic and the p boundaries are closed).

Kinetic energy balance equation – eddy

Similarly, we can obtain a balance equation for the eddy kinetic energy terms $[A^{*2}]$:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\left[A^{*2}\right]\right) = -\frac{\partial}{R_E \partial \phi}\left([v]\frac{1}{2}\left[A^{*2}\right]\right) - \frac{\partial}{\partial p}\left([\omega]\frac{1}{2}\left[A^{*2}\right]\right) \qquad \text{flux divergence terms}$$

$$+ [A^* S^*] \qquad \text{sources/sinks}$$

$$- [v^* A^*]\frac{\partial [A]}{R_E \partial \phi} - [\omega^* A^*]\frac{\partial [A]}{\partial p} \qquad \text{conversion terms} \qquad (2.124)$$

In the derivation, we have neglected terms higher than second order in deviation from the zonal mean

Kinetic energy balance equation

Let us know calculate the balance equation for the mean kinetic energy

$$K_M = \frac{1}{2} \int \left([u]^2 + [v]^2 \right) \, dm \tag{2.125}$$

Consider the balance equations for u and v and note that Coriolis, pressure-gradient, and frictional terms are source/sink terms in the language of (2.119):

$$\frac{\partial u}{\partial t} = -\frac{u}{R_E \cos \phi} \frac{\partial u}{\partial \lambda} - \frac{v}{R_E} \frac{\partial u}{\partial \phi} - \omega \frac{\partial u}{\partial p} + \frac{\tan \phi}{R_E} uv + fv - \frac{g}{R_E \cos \phi} \frac{\partial z}{\partial \lambda} + F_{r,\lambda} \tag{2.126}$$

$$\frac{\partial v}{\partial t} = -\frac{u}{R_E \cos \phi} \frac{\partial v}{\partial \lambda} - \frac{v}{R_E} \frac{\partial v}{\partial \phi} - \omega \frac{\partial v}{\partial p} + \frac{\tan \phi}{R_E} u^2 - fu - \frac{g}{R_E} \frac{\partial z}{\partial \phi} + F_{r,\phi} \tag{2.127}$$

Applying the formalism of the preceding slides, we find that the advection terms in (2.126) and (2.127) yield flux divergence terms in (2.123); the Coriolis terms cancel; the remaining terms are (taking integrals over the entire atmosphere)

$$C(K_E, K_M) = \int \left([v^* u^*] \frac{\partial [u]}{R_E \partial \phi} + [\omega^* u^*] \frac{\partial [u]}{\partial p} + [v^* v^*] \frac{\partial [v]}{R_E \partial \phi} + [\omega^* v^*] \frac{\partial [v]}{\partial p} - [v] \frac{\tan \phi}{R_E} [u^* u^*] \right) \, dm \tag{2.128}$$

$$C(P_M, K_M) = -\int \left([v] \, g \frac{\partial [z]}{R_E \partial \phi} \right) \, dm = -\int [\omega] \, [\alpha] \, dm \tag{2.129}$$

$$D(K_M) = -\int \left([u] \, [F_\lambda] + [v] \, [F_\phi] \right) \, dm \tag{2.130}$$

Kinetic energy balance equations

Mean kinetic energy

The mean kinetic energy balance equation is

$$\frac{\partial K_M}{\partial t} = C\left(P_M, K_M\right) + C\left(K_E, K_M\right) - D\left(K_M\right) \tag{2.131}$$

where the terms represent conversion of eddy kinetic energy to mean kinetic energy, conversion of mean potential to kinetic energy (through vertical motion), and dissipation of mean kinetic energy (by friction).

Eddy kinetic energy

The eddy kinetic energy balance equation is

$$\frac{\partial K_E}{\partial t} = C\left(P_E, K_E\right) - C\left(K_E, K_M\right) - D\left(K_E\right) \tag{2.132}$$

$$C\left(K_E, K_M\right) = \text{as in (2.128)}$$

$$C\left(P_E, K_E\right) = -\int \left[\omega^* \alpha^*\right] \, dm \tag{2.133}$$

$$D\left(K_E\right) = -\int \left(\left[u^* F_\lambda^*\right] + \left[v^* F_\phi^*\right]\right) \, dm \tag{2.134}$$

where the terms represent conversion of eddy kinetic energy to mean kinetic energy, conversion of eddy potential to kinetic energy (through vertical motion), and dissipation of eddy kinetic energy (by friction).

Mean potential energy balance equation

Recall the mean potential energy definition (2.118):

$$P_M = \frac{c_p}{2} \int \gamma \left([\bar{T}] - \overline{\tilde{T}} \right)^2 \, dm \tag{2.135}$$

Beginning with the first law of thermodynamics

$$\frac{\partial T}{\partial t} = -\vec{V} \cdot \nabla T - \omega \frac{\partial T}{\partial p} + \frac{\omega \alpha}{c_p} + \frac{Q}{c_p} = -\vec{V} \cdot \nabla T - \omega \frac{T}{\Theta} \frac{\partial \Theta}{\partial p} + \frac{Q}{c_p}, \tag{2.136}$$

we can derive a balance equation for P_M by multiplying (2.136) by $\gamma([T] - \tilde{T})$, averaging zonally and temporally, and integrating over the entire atmosphere:

$$\frac{\partial P_M}{\partial t} = G(P_M) - C(P_M, P_E) - C(P_M, K_M), \quad \text{where} \tag{2.137}$$

$$G(P_M) = \int \gamma \left([T] - \tilde{T} \right) \left([Q] - \tilde{Q} \right) \, dm \tag{2.138}$$

$$C(P_M, P_E) = -c_p \int \left(\gamma [v^* T^*] \frac{\partial [T]}{R_E \partial \phi} + p^{-\kappa} [\omega^* T^*] \frac{\partial}{\partial p} \left[\gamma p^{\kappa} \left([T] - \tilde{T} \right) \right] \right) \, dm \tag{2.139}$$

$$C(P_M, K_M) = \text{as in (2.129)}$$

Generation of mean potential energy is by the excess heating in the tropics; conversion of mean to eddy potential energy is by down-gradient eddy transport of latent heat.

Eddy potential energy balance equation

By a similar procedure, we can use (2.136) to derive a balance equation for the eddy potential energy

$$P_E = \frac{c_p}{2} \int \gamma \left[\overline{T'^2} \right] \, dm + \frac{c_p}{2} \int \gamma \left[\overline{(T^*)^2} \right] \, dm \tag{2.140}$$

which takes the form

$$\frac{\partial P_E}{\partial t} = G(P_E) + C(P_M, P_E) - C(P_E, K_E), \quad \text{where} \tag{2.141}$$

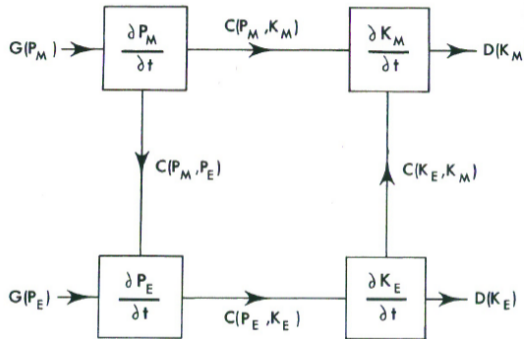$$G(P_E) = \int \gamma \left[T^* Q^* \right] \, dm \tag{2.142}$$

$$C(P_M, P_E) = \text{as in (2.139)}$$

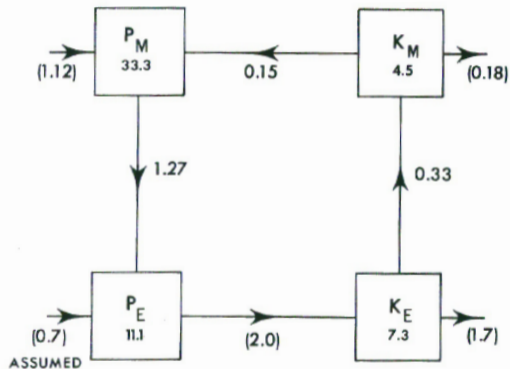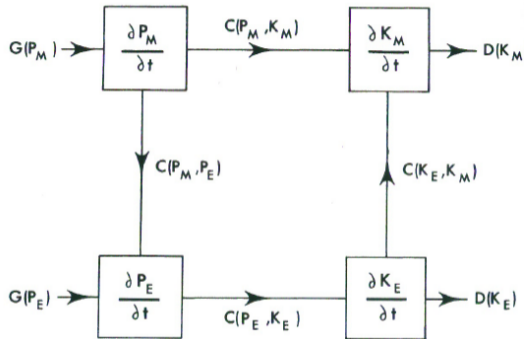$$C(P_E, K_E) = \text{as in (2.133)}$$

Eddy potential energy is generated by heating of warm anomalies or cooling of cold anomalies.
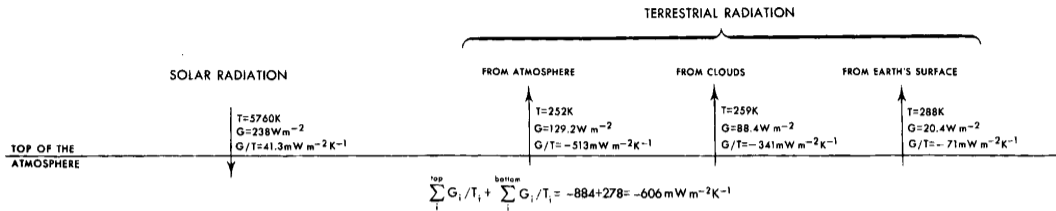
Lorenz cycle

Lorenz cycle

Entropy cycle

Entropy generation by irreversible processes

Irreversible processes increase entropy. There are many such irreversible processes in the climate system: absorption of radiation, turbulent dissipation of kinetic energy, precipitation, . . . Eventually, these irreversible processes would lead to a highly disordered state where the general circulation has been dissipated by friction, the hydrological cycle has ceased, etc. How does the climate system maintain its state of high organization?

The answer is that the climate system is not isolated from the environment, so that entropy transfer can occur across the boundaries. Thus, the condition for maintaining constant entropy in the atmosphere is that the entropy fluxes (F_i/T_i) across the boundaries balance the entropy production terms (σ_i) within the atmosphere:

$$\frac{F_{\text{rad}}}{T_{\text{rad}}} + \frac{F_{\text{SH}}}{T_{\text{SH}}} + \sigma_{\text{rad}} + \sigma_{\text{LH}} + \sigma_{\text{SH}} + \sigma_{\text{dis}} = 0 \qquad (2.143)$$

Entropy cycle

TERRESTRIAL RADIATION

| SOLAR RADIATION | FROM ATMOSPHERE | FROM CLOUDS | FROM EARTH'S SURFACE |

TOP OF THE ATMOSPHERE

SOLAR RADIATION
$T=5760K$
$G=238 W m^{-2}$
$G/T=41.3 mW m^{-2} K^{-1}$

FROM ATMOSPHERE
$T=252K$
$G=129.2 W m^{-2}$
$G/T=-513 mW m^{-2} K^{-1}$

FROM CLOUDS
$T=259K$
$G=88.4 W m^{-2}$
$G/T=-341 mW m^{-2} K^{-1}$

FROM EARTH'S SURFACE
$T=288K$
$G=20.4 W m^{-2}$
$G/T=-71 mW m^{-2} K^{-1}$

$$\sum_i^{top} G_i/T_i + \sum_i^{bottom} G_i/T_i = -884+278 = -606 \, mW \, m^{-2} K^{-1}$$

ENTROPY GENERATION PROCESSES

ABSORPTION SOLAR RADIATION
$T=252K$
$\rho Q=68 W m^{-2}$
$\sigma=258 mW m^{-2} K^{-1}$

ABSORPTION TERRESTRIAL RADIATION
$T=252K$
$\rho Q=47.6 W m^{-2}$
$\sigma=24 mW m^{-2} K^{-1}$

LATENT HEAT RELEASE
$T=266K$
$\rho Q=79 W m^{-2}$
$\sigma=298 mW m^{-2} K^{-1}$

SENSIBLE HEATING
$T=280K$
$\rho Q=20.4 W m^{-2}$
$\sigma=2.1 mW m^{-2} K^{-1}$

FRICTIONAL HEATING
$T=280K$
$\rho Q=1.9 W m^{-2}$
$\sigma=7 mW m^{-2} K^{-1}$

$$\sum_i \sigma_i = 589 \, mW \, m^{-2} K^{-1}$$

EARTH'S SURFACE

SOLAR RADIATION
$T=5760K$
$G=170 W m^{-2}$
$G/T=-29.5 mW m^{-2} K^{-1}$

SENSIBLE HEAT
$T=288K$
$G=20.4 W m^{-2}$
$G/T=71 mW m^{-2} K^{-1}$

TERRESTRIAL RADIATION
$T=288K$
$G=68 W m^{-2}$
$G/T=236 mW m^{-2} K^{-1}$