Formale Grundlagen (Logik)
Modul 04-006-1001

Statement Logic IV

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(Slides by Imke Driemel & Sandhya Sundaresan,
based on Partee, ter Meulen und Wall 1990
"Mathematical Methods in Linguistics")

Recap: Formal components of a proof

- constructing a proof/argument consists of two parts
  1. a number of statements, called **premises**: these are just statements that we, for the sake of argument, assume to be True
  2. a **conclusion**, whose truth is demonstrated to necessarily follow from the assumed truth of the premises

(1)
$$\begin{array}{ll} & \text{premise 1} \\ & \text{premise 2} \\ & \ldots \\ \hline \therefore & \text{conclusion} \end{array}$$

- a proof is called **valid** iff there is no uniform assignment of truth values to its atomic statements which makes all its premises true and its conclusion false
- a proof is called **invalid** iff there is at least one uniform assignment of truth values to its atomic statements which makes all its premises true and its conclusion false
- premises and conclusion of a proof are related by the conditional $\rightarrow$ (premise 1 $\wedge$ premise 2 $\wedge \ldots \rightarrow$ conclusion)

Recap: Proofs

- let us remind ourselves of this based on the proof $\boxed{\text{Modus Tollens}}$

(2)

$$\begin{array}{ll} & \text{If Jack drinks beer, he will get drunk.} \\ & \text{Jack doesn't get drunk.} \\ \hline \therefore & \text{Jack doesn't drink beer.} \end{array}$$

(3)

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

- we show the validity of the proof with a truth table (all values are True in the last column)

(4)

p	q	$(p \rightarrow q)$	$((p \rightarrow q) \wedge (\neg q))$	$(((p \rightarrow q) \wedge (\neg q)) \rightarrow (\neg p))$
1	1	1	0	1
1	0	0	0	1
0	1	1	0	1
0	0	1	1	1

Recap: Fallacies (invalid proofs)

- example of an invalid proof: **fallacy of affirming the consequent**

(5) If Marie eats another pizza, she will get sick.
Marie gets sick.
$\therefore\!\!\!/$ Marie eats another pizza.

(6)
$$\begin{array}{ll} & (p \rightarrow q) \\ & q \\ \hline \therefore\!\!\!/ & p \end{array}$$

- we can show that the proof is invalid with a truth table (not all values are True in the last column)

(7)

p	q	$(p \rightarrow q)$	$((p \rightarrow q) \wedge q)$	$(((p \rightarrow q) \wedge q) \rightarrow p)$
1	1	1	1	1
1	0	0	0	1
0	1	1	1	0
0	0	1	0	1

Simple proofs

(8) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(9) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(10) **Hypothetical Syllogism**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(11) **Disjunctive Syllogism**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(12) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(13) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(14) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

Simple proofs

(8) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(9) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(10) **Hypothetical Syllogism**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(11) **Disjunctive Syllogism**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(12) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(13) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(14) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

Given the premises 1.-5. we can prove t !

(15) *simple proof*:

1. $(p \rightarrow q)$
2. $(p \vee s)$
3. $(q \rightarrow r)$
4. $(s \rightarrow t)$
5. $(\neg r)$
6.
7.
8.
9. t

Simple proofs

(8) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(9) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(10) **Hypothetical Syllogism**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(11) **Disjunctive Syllogism**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(12) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(13) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(14) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

Given the premises 1.-5. we can prove t !

(15) *simple proof*:

1.	$(p \rightarrow q)$	
2.	$(p \vee s)$	
3.	$(q \rightarrow r)$	
4.	$(s \rightarrow t)$	
5.	$(\neg r)$	
6.	$(\neg q)$	3,5 MT
7.	$(\neg p)$	1,6 MT
8.	s	2,7 DS
9.	t	4,8 MP

Simple proofs: Reverse engineering

(16) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(17) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(18) **Hypothetical Syllogism**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(19) **Disjunctive Syllogism**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(20) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(21) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(22) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

t is part of 4.
With which proof can we move from 4. to t?

Given the premises 1.-5. we can prove t!

(23) *simple proof*:

1. $(p \rightarrow q)$
2. $(p \vee s)$
3. $(q \rightarrow r)$
4. $(s \rightarrow t)$
5. $(\neg r)$
6.
7.
8.
9. t

Simple proofs: Reverse engineering

(16) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(17) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(18) **Hypothetical Syllogism**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(19) **Disjunctive Syllogism**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(20) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(21) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(22) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

t is part of 4.
With which proof can we move from 4. to t?
With MP, but only if s is true!

Given the premises 1.-5. we can prove t !

(23) *simple proof*:

1. $(p \rightarrow q)$
2. $(p \vee s)$
3. $(q \rightarrow r)$
4. $(s \rightarrow t)$
5. $(\neg r)$
6.
7.
8. s
9. t 4,8 MP

Simple proofs: Reverse engineering

(24) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(25) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(26) **Hypothetical Syllogism**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(27) **Disjunctive Syllogism**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(28) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(29) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(30) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

s is part of 2.
With which proof can we move from 2. to s?

Given the premises 1.-5. we can prove t !

(31) *simple proof*:

1. $(p \rightarrow q)$
2. $(p \vee s)$
3. $(q \rightarrow r)$
4. $(s \rightarrow t)$
5. $(\neg r)$
6.
7.
8. s
9. t 4,8 MP

Simple proofs: Reverse engineering

(24) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(25) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(26) **Hypothetical Syllogism**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(27) **Disjunctive Syllogism**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(28) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(29) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(30) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

s is part of 2.
With which proof can we move from 2. to s?
With DS, but only if $(\neg p)$ is true!

Given the premises 1.-5. we can prove t !

(31) *simple proof*:

1. $(p \rightarrow q)$
2. $(p \vee s)$
3. $(q \rightarrow r)$
4. $(s \rightarrow t)$
5. $(\neg r)$
6.
7. $(\neg p)$
8. s — 2,7 DS
9. t — 4,8 MP

Simple proofs: Reverse engineering

(32) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(33) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(34) **Hypothetical Syllogism**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(35) **Disjunctive Syllogism**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(36) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(37) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(38) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

p is part of 1. and 2.
With which proof can we move from either 2. or 1. to $(\neg p)$?

Given the premises 1.-5. we can prove t !

(39) *simple proof*:

1.	$(p \rightarrow q)$	
2.	$(p \vee s)$	
3.	$(q \rightarrow r)$	
4.	$(s \rightarrow t)$	
5.	$(\neg r)$	
6.		
7.	$(\neg p)$	
8.	s	2,7 DS
9.	t	4,8 MP

Simple proofs: Reverse engineering

(32) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(33) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(34) **Hypothetical Syllogism**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(35) **Disjunctive Syllogism**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(36) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(37) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(38) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

p is part of 1. and 2.
With which proof can we move from either 2. or 1. to $(\neg p)$?
With MT, but only if $(\neg q)$ is true!

Given the premises 1.-5. we can prove t !

(39) *simple proof*:

1.	$(p \rightarrow q)$	
2.	$(p \vee s)$	
3.	$(q \rightarrow r)$	
4.	$(s \rightarrow t)$	
5.	$(\neg r)$	
6.	$(\neg q)$	
7.	$(\neg p)$	1,6 MT
8.	s	2,7 DS
9.	t	4,8 MP

Simple proofs: Reverse engineering

(40) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(41) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(42) **Hypothetical Syllogism**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(43) **Disjunctive Syllogism**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(44) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(45) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(46) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

Does $(\neg q)$ *follow from any of the premises 1.-5.?*

Given the premises 1.-5. we can prove t !

(47) *simple proof*:

1.	$(p \rightarrow q)$	
2.	$(p \vee s)$	
3.	$(q \rightarrow r)$	
4.	$(s \rightarrow t)$	
5.	$(\neg r)$	
6.	$(\neg q)$	
7.	$(\neg p)$	1,6 MT
8.	s	2,7 DS
9.	t	4,8 MP

Simple proofs: Reverse engineering

(40) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(41) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(42) **Hypothetical Syllogism**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(43) **Disjunctive Syllogism**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(44) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(45) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(46) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

Does $(\neg q)$ *follow from any of the premises 1.-5.?*
Yes, with MT, we can conclude $(\neg q)$ *from 3. and 5.!*

Given the premises 1.-5. we can prove t !

(47) *simple proof*:

1. $(p \rightarrow q)$
2. $(p \vee s)$
3. $(q \rightarrow r)$
4. $(s \rightarrow t)$
5. $(\neg r)$
6. $(\neg q)$ 3,5 MT
7. $(\neg p)$ 1,6 MT
8. s 2,7 DS
9. t 4,8 MP

Complex proofs

(48) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(49) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(50) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(51) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(52) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(53) **Identity Laws**:

a. $x \vee False \Leftrightarrow x$

b. $x \wedge False \Leftrightarrow False$

c. $x \vee True \Leftrightarrow True$

d. $x \wedge True \Leftrightarrow x$

(54) **Conditional Laws**:

a. $(p \rightarrow q) \Leftrightarrow ((\neg p) \vee q)$

b. $(p \rightarrow q) \Leftrightarrow ((\neg q) \rightarrow (\neg p))$

(55) **Commutative Laws**:

a. $x \vee y \Leftrightarrow y \vee x$

b. $x \wedge y \Leftrightarrow y \wedge x$

(56) **Associative Laws**:

a. $(x \vee y) \vee z \Leftrightarrow x \vee (y \vee z)$

b. $(x \wedge y) \wedge z \Leftrightarrow x \wedge (y \wedge z)$

Given the premises 1.-2. we can prove $(p \rightarrow q)$!

(57) *complex proof*:

1.	$(p \rightarrow (q \vee r))$	
2.	$(\neg r)$	
3.	$((\neg p) \vee (q \vee r))$	1 Cond
4.	$(((\neg p) \vee q) \vee r)$	3 Ass
5.	$(((\neg p) \vee q) \vee F)$	4 Neg
6.	$(((\neg p) \vee q))$	5 Ident
7.	$(p \rightarrow q)$	6 Cond

Complex proofs: Reverse engineering

(58) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(59) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(60) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(61) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(62) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(63) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(64) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

Is there a proof such that we can conclude $(p \rightarrow q)$ given our premises?

Given the premises 1.-2. we can prove $(p \rightarrow q)$!

(65) *complex proof*:

1. $(p \rightarrow (q \vee r))$
2. $(\neg r)$
3.
4.
5.
6.
7. $(p \rightarrow q)$

Complex proofs: Reverse engineering

(58) **Modus Ponens**

$$\begin{array}{ll} & (p \to q) \\ & p \\ \hline \therefore & q \end{array}$$

(59) **Modus Tollens**

$$\begin{array}{ll} & (p \to q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(60) **Hyp. Syll.**

$$\begin{array}{ll} & (p \to q) \\ & (q \to r) \\ \hline \therefore & (p \to r) \end{array}$$

(61) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(62) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(63) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(64) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

Is there a proof such that we can conclude $(p \to q)$ given our premises?
Not really . . . we have to make use of equivalence laws to substitute our conclusion into a formula we can work with.

Given the premises 1.-2. we can prove $(p \to q)$!

(65) *complex proof*:

1. $(p \to (q \vee r))$
2. $(\neg r)$
3.
4.
5.
6.
7. $(p \to q)$

Complex proofs: Reverse engineering

(66) **Idempotent Laws**:

a. $(x \vee x) \Leftrightarrow x$

b. $(x \wedge x) \Leftrightarrow x$

(67) **Complement Laws**:

a. $p \vee (\neg p) \Leftrightarrow True$

b. $\neg(\neg p) \Leftrightarrow p$

c. $p \wedge (\neg p) \Leftrightarrow False$

(68) **Identity Laws**:

a. $x \vee False \Leftrightarrow x$

b. $x \wedge False \Leftrightarrow False$

c. $x \vee True \Leftrightarrow True$

d. $x \wedge True \Leftrightarrow x$

(69) **Commutative Laws**:

a. $x \vee y \Leftrightarrow y \vee x$

b. $x \wedge y \Leftrightarrow y \wedge x$

(70) **Distributive Laws**:

a. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

b. $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

(71) **DeMorgan's Laws**:

a. $\neg(p \vee q) \Leftrightarrow (\neg p) \wedge (\neg q)$

b. $\neg(p \wedge q) \Leftrightarrow (\neg p) \vee (\neg q)$

(72) **Conditional Laws**:

a. $(p \rightarrow q) \Leftrightarrow ((\neg p) \vee q)$

b. $(p \rightarrow q) \Leftrightarrow ((\neg q) \rightarrow (\neg p))$

(73) **Associative Laws**:

a. $(x \vee y) \vee z \Leftrightarrow x \vee (y \vee z)$

b. $(x \wedge y) \wedge z \Leftrightarrow x \wedge (y \wedge z)$

Which laws can we apply to $(p \rightarrow q)$*?*

Given the premises 1.-2. we can prove $(p \rightarrow q)$!

(74) *complex proof*:

1. $(p \rightarrow (q \vee r))$
2. $(\neg r)$
3.
4.
5.
6.
7. $(p \rightarrow q)$

Complex proofs: Reverse engineering

(66) **Idempotent Laws**:

a. $(x \vee x) \Leftrightarrow x$

b. $(x \wedge x) \Leftrightarrow x$

(67) **Complement Laws**:

a. $p \vee (\neg p) \Leftrightarrow True$

b. $\neg(\neg p) \Leftrightarrow p$

c. $p \wedge (\neg p) \Leftrightarrow False$

(68) **Identity Laws**:

a. $x \vee False \Leftrightarrow x$

b. $x \wedge False \Leftrightarrow False$

c. $x \vee True \Leftrightarrow True$

d. $x \wedge True \Leftrightarrow x$

(69) **Commutative Laws**:

a. $x \vee y \Leftrightarrow y \vee x$

b. $x \wedge y \Leftrightarrow y \wedge x$

(70) **Distributive Laws**:

a. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

b. $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

(71) **DeMorgan's Laws**:

a. $\neg(p \vee q) \Leftrightarrow (\neg p) \wedge (\neg q)$

b. $\neg(p \wedge q) \Leftrightarrow (\neg p) \vee (\neg q)$

(72) **Conditional Laws**:

a. $(p \rightarrow q) \Leftrightarrow ((\neg p) \vee q)$

b. $(p \rightarrow q) \Leftrightarrow ((\neg q) \rightarrow (\neg p))$

(73) **Associative Laws**:

a. $(x \vee y) \vee z \Leftrightarrow x \vee (y \vee z)$

b. $(x \wedge y) \wedge z \Leftrightarrow x \wedge (y \wedge z)$

Which laws can we apply to $(p \rightarrow q)$?
Conditional laws! The first one seems more promising . . .

Given the premises 1.-2. we can prove $(p \rightarrow q)$!

(74) *complex proof*:

1. $(p \rightarrow (q \vee r))$
2. $(\neg r)$
3.
4.
5.
6. $((\neg p) \vee q)$
7. $(p \rightarrow q)$ 6 Cond

Complex proofs: Reverse engineering

(75) **Modus Ponens**

$$\begin{array}{ll} & (p \to q) \\ & p \\ \hline \therefore & q \end{array}$$

(76) **Modus Tollens**

$$\begin{array}{ll} & (p \to q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(77) **Hyp. Syll.**

$$\begin{array}{ll} & (p \to q) \\ & (q \to r) \\ \hline \therefore & (p \to r) \end{array}$$

(78) **Dis. Syll.**

$$\begin{array}{ll} & (p \lor q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(79) **Simplification**

$$\begin{array}{ll} & (p \land q) \\ \hline \therefore & p \end{array}$$

(80) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \land q) \end{array}$$

(81) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \lor q) \end{array}$$

The next proof is difficult to see. We will derive 6. by applying DS with the help of premise 2. What would be the other statement we have to create?

Given the premises 1.-2. we can prove $(p \to q)$!

(82) *complex proof*:

1. $(p \to (q \lor r))$
2. $(\neg r)$
3.
4.
5.
6. $((\neg p) \lor q)$
7. $(p \to q)$ 6 Cond

Complex proofs: Reverse engineering

(75) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(76) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(77) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(78) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(79) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(80) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(81) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

The next proof is difficult to see. We will derive 6. by applying DS with the help of premise 2. What would be the other statement we have to create?
We have to create the disjunction which is the first premise of DS!

Given the premises 1.-2. we can prove $(p \rightarrow q)$!

(82) *complex proof*:

1. $(p \rightarrow (q \vee r))$
2. $(\neg r)$
3.
4.
5. $(r \vee ((\neg p) \vee q))$
6. $((\neg p) \vee q)$ 2,5 DS
7. $(p \rightarrow q)$ 6 Cond

Complex proofs: Reverse engineering

(83) **Idempotent Laws**:

a. $(x \vee x) \Leftrightarrow x$

b. $(x \wedge x) \Leftrightarrow x$

(84) **Complement Laws**:

a. $p \vee (\neg p) \Leftrightarrow \mathit{True}$

b. $\neg(\neg p) \Leftrightarrow p$

c. $p \wedge (\neg p) \Leftrightarrow \mathit{False}$

(85) **Identity Laws**:

a. $x \vee \mathit{False} \Leftrightarrow x$

b. $x \wedge \mathit{False} \Leftrightarrow \mathit{False}$

c. $x \vee \mathit{True} \Leftrightarrow \mathit{True}$

d. $x \wedge \mathit{True} \Leftrightarrow x$

(86) **Commutative Laws**:

a. $x \vee y \Leftrightarrow y \vee x$

b. $x \wedge y \Leftrightarrow y \wedge x$

(87) **Distributive Laws**:

a. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

b. $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

(88) **DeMorgan's Laws**:

a. $\neg(p \vee q) \Leftrightarrow (\neg p) \wedge (\neg q)$

b. $\neg(p \wedge q) \Leftrightarrow (\neg p) \vee (\neg q)$

(89) **Conditional Laws**:

a. $(p \rightarrow q) \Leftrightarrow ((\neg p) \vee q)$

b. $(p \rightarrow q) \Leftrightarrow ((\neg q) \rightarrow (\neg p))$

(90) **Associative Laws**:

a. $(x \vee y) \vee z \Leftrightarrow x \vee (y \vee z)$

b. $(x \wedge y) \wedge z \Leftrightarrow x \wedge (y \wedge z)$

How does that help us?

Given the premises 1.-2. we can prove $(p \rightarrow q)$!

(91) *complex proof*:

1. $(p \rightarrow (q \vee r))$
2. $(\neg r)$
3.
4.
5. $(r \vee ((\neg p) \vee q))$
6. $((\neg p) \vee q)$ 2,5 DS
7. $(p \rightarrow q)$ 6 Cond

Complex proofs: Reverse engineering

(83) **Idempotent Laws**:

a. $(x \vee x) \Leftrightarrow x$

b. $(x \wedge x) \Leftrightarrow x$

(84) **Complement Laws**:

a. $p \vee (\neg p) \Leftrightarrow True$

b. $\neg(\neg p) \Leftrightarrow p$

c. $p \wedge (\neg p) \Leftrightarrow False$

(85) **Identity Laws**:

a. $x \vee False \Leftrightarrow x$

b. $x \wedge False \Leftrightarrow False$

c. $x \vee True \Leftrightarrow True$

d. $x \wedge True \Leftrightarrow x$

(86) **Commutative Laws**:

a. $x \vee y \Leftrightarrow y \vee x$

b. $x \wedge y \Leftrightarrow y \wedge x$

(87) **Distributive Laws**:

a. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

b. $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

(88) **DeMorgan's Laws**:

a. $\neg(p \vee q) \Leftrightarrow (\neg p) \wedge (\neg q)$

b. $\neg(p \wedge q) \Leftrightarrow (\neg p) \vee (\neg q)$

(89) **Conditional Laws**:

a. $(p \rightarrow q) \Leftrightarrow ((\neg p) \vee q)$

b. $(p \rightarrow q) \Leftrightarrow ((\neg q) \rightarrow (\neg p))$

(90) **Associative Laws**:

a. $(x \vee y) \vee z \Leftrightarrow x \vee (y \vee z)$

b. $(x \wedge y) \wedge z \Leftrightarrow x \wedge (y \wedge z)$

How does that help us?
It turns out that we can simplify 1. to 5.!

Given the premises 1.-2. we can prove $(p \rightarrow q)$!

(91) *complex proof*:

1. $(p \rightarrow (q \vee r))$
2. $(\neg r)$
3. $((\neg p) \vee (q \vee r))$ 1 Cond
4. $(((\neg p) \vee q) \vee r)$ 3 Ass
5. $(r \vee ((\neg p) \vee q))$ 4 Comm
6. $((\neg p) \vee q)$ 2,5 DS
7. $(p \rightarrow q)$ 6 Cond

Direct conditional proofs

- complex proofs such as the one we just went through are tricky
- fortunately, there is a simpler method for such a proof, i.e. a proof where the conclusion is a conditional
- if the conclusion of a proof contains a conditional as the main connective, we can use a method of argumentation called **conditional proof**
  - suppose a proof has premises: $p_1, p_2, \ldots p_n$ and $(q \rightarrow r)$ as the conclusion
  - in the conditional proof, we add the antecedent q of the conclusion as an additional **auxiliary premise**
  - we then derive r from the premises $p_1, p_2, \ldots p_n$ and the auxiliary premise q
- the validity of the conditional proof is based on the following logical equivalence:

(92) $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow (q \rightarrow r) \Leftrightarrow (p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge q) \rightarrow r$

Direct conditional proofs

- a conditional proof begins with the assumption that the antecedent is true, then logical reasoning is used to show that the consequent must also be true (given other premises)
- a conditional proof shows that the antecedent implies the consequent

Given the premises 1.-2. we can prove $(p \rightarrow q)$!

(93) *complex proof*:

1.	$(p \rightarrow (q \vee r))$	
2.	$(\neg r)$	
3.	$((\neg p) \vee (q \vee r))$	1 Cond
4.	$(((\neg p) \vee q) \vee r)$	3 Ass
5.	$(r \vee ((\neg p) \vee q))$	4 Comm
6.	$((\neg p) \vee q)$	2,5 DS
7.	$(p \rightarrow q)$	6 Cond

Given the premises 1.-2. we can prove $(p \rightarrow q)$!

(94) *conditional proof*:

1.	$(p \rightarrow (q \vee r))$	
2.	$(\neg r)$	
3.	$\mid\ p$	Aux
4.	$\mid$	
5.	$\mid$	
6.	$\mid\ q$	
7.	$(p \rightarrow q)$	3-6 CP

Direct conditional proofs

(95) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(96) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(97) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(98) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(99) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(100) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(101) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

What are the next steps?

Given the premises 1.-2. we can prove $(p \rightarrow q)$!

(102) *conditional proof*:

$$\begin{array}{lll} 1. & (p \rightarrow (q \vee r)) & \\ 2. & (\neg r) & \\ 3. & \quad \vert\ p & \text{Aux} \\ 4. & \quad \vert & \\ 5. & \quad \vert & \\ 6. & \quad \vert\ q & \\ 7. & (p \rightarrow q) & \text{3-6 CP} \end{array}$$

Direct conditional proofs

(95) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(96) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(97) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(98) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(99) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(100) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(101) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

What are the next steps?

Given the premises 1.-2. we can prove $(p \rightarrow q)$!

(102) *conditional proof*:

$$\begin{array}{lll} 1. & (p \rightarrow (q \vee r)) & \\ 2. & (\neg r) & \\ 3. & \quad| \; p & \text{Aux} \\ 4. & \quad| \; (q \vee r) & \text{1,3 MP} \\ 5. & \quad| & \\ 6. & \quad| \; q & \\ 7. & (p \rightarrow q) & \text{3-6 CP} \end{array}$$

Direct conditional proofs

(95) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(96) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(97) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(98) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(99) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(100) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(101) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

What are the next steps?

Given the premises 1.-2. we can prove $(p \rightarrow q)$!

(102) *conditional proof*:

$$\begin{array}{lll} 1. & (p \rightarrow (q \vee r)) & \\ 2. & (\neg r) & \\ 3. & \quad | \; p & \text{Aux} \\ 4. & \quad | \; (q \vee r) & \text{1,3 MP} \\ 5. & \quad | \; (r \vee q) & \text{4 Comm} \\ 6. & \quad | \; q & \\ 7. & (p \rightarrow q) & \text{3-6 CP} \end{array}$$

Direct conditional proofs

(95) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(96) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(97) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(98) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(99) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(100) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(101) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

What are the next steps?

Given the premises 1.-2. we can prove $(p \rightarrow q)$!

(102) *conditional proof*:

$$\begin{array}{lll} 1. & (p \rightarrow (q \vee r)) & \\ 2. & (\neg r) & \\ 3. & \quad\mid\; p & \text{Aux} \\ 4. & \quad\mid\; (q \vee r) & \text{1,3 MP} \\ 5. & \quad\mid\; (r \vee q) & \text{4 Comm} \\ 6. & \quad\mid\; q & \text{2,5 DS} \\ 7. & (p \rightarrow q) & \text{3-6 CP} \end{array}$$

Direct conditional proofs

- conditional proofs are called conditional because they rely on an additional premise
- the conclusion is true under the condition that the auxiliary premise is true

Given the premises 1.-2. we can prove $(p \rightarrow q)$!

(103) *complex proof*:

1.	$(p \rightarrow (q \vee r))$	
2.	$(\neg r)$	
3.	$((\neg p) \vee (q \vee r))$	1 Cond
4.	$(((\neg p) \vee q) \vee r)$	3 Ass
5.	$(r \vee ((\neg p) \vee q))$	4 Comm
6.	$((\neg p) \vee q)$	2,5 DS
7.	$(p \rightarrow q)$	6 Cond

Given the premises 1.-2. we can prove $(p \rightarrow q)$!

(104) *conditional proof*:

1.	$(p \rightarrow (q \vee r))$	
2.	$(\neg r)$	
3.	p	Aux
4.	$(q \vee r)$	1,3 MP
5.	$(r \vee q)$	4 Comm
6.	q	2,5 DS
7.	$(p \rightarrow q)$	3-6 CP

Direct conditional proofs

- conditional proofs are called conditional because they rely on additional premise
- the conclusion is true under the condition that the auxiliary assumption is true

  - we indicate with a vertical bar every line of the proof which is based on the auxiliary premise
  - this is to remind ourselves that we are working with an additional special assumption
  - it is very important to always cancel that auxiliary premise by the rule of Conditional Proof before ending the entire proof (means going back to original position)
  - under the assumption that p is true, the conditional $(p \rightarrow q)$ holds
  - $(p \rightarrow q)$ does not follow directly from premises 1. and 2.

Given the premises 1.-2. we can prove $(p \rightarrow q)$!

(105) *conditional proof*:

1.	$(p \rightarrow (q \vee r))$	
2.	$(\neg r)$	
3.	| p	Aux
4.	| $(q \vee r)$	1,3 MP
5.	| $(r \vee q)$	4 Comm
6.	| q	2,5 DS
7.	$(p \rightarrow q)$	3-6 CP

Nested conditional proofs

- a conditional proof can be more complicated with two levels of embedding

  - two auxiliary premises = two vertical bars, one inside the other where the inner bar depends on the outer bar
  - we can always use a statement from a higher level in a lower level
  - but we may not use a statement from a lower level in a higher level
  - under the assumption that p is true, the conditional $(p \to s)$ holds
  - under the assumption that $(q \to s)$ is true, the conditional $((q \to s) \to (p \to s))$ holds

Given the premise 1. we can prove $((q \to s) \to (p \to s))$!

(106) *conditional proof*:

1.	$(p \to (q \wedge r))$	
2.	$\quad\vert\ (q \to s)$	Aux
3.	$\quad\vert\quad\vert\ p$	Aux
4.	$\quad\vert\quad\vert$	
5.	$\quad\vert\quad\vert$	
6.	$\quad\vert\quad\vert\ s$	
7.	$\quad\vert\ (p \to s)$	3-6 CP
8.	$((q \to s) \to (p \to s))$	2-7 CP

Nested conditional proofs

(107) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(108) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(109) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(110) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(111) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(112) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(113) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

What are the next steps?

Given the premise 1. we can prove $((q \rightarrow s) \rightarrow (p \rightarrow s))$!

(114) *conditional proof*:

1.	$(p \rightarrow (q \wedge r))$	
2.	$\mid (q \rightarrow s)$	Aux
3.	$\mid \mid p$	Aux
4.	$\mid \mid$	
5.	$\mid \mid$	
6.	$\mid \mid s$	
7.	$\mid (p \rightarrow s)$	3-6 CP
8.	$((q \rightarrow s) \rightarrow (p \rightarrow s))$	2-7 CP

Nested conditional proofs

(107) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(108) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(109) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(110) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(111) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(112) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(113) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

What are the next steps?

Given the premise 1. we can prove $((q \rightarrow s) \rightarrow (p \rightarrow s))$!

(114) *conditional proof*:

1.	$(p \rightarrow (q \wedge r))$	
2.	$\quad\mid (q \rightarrow s)$	Aux
3.	$\quad\mid\quad\mid p$	Aux
4.	$\quad\mid\quad\mid (q \wedge r)$	1,3 MP
5.	$\quad\mid\quad\mid$	
6.	$\quad\mid\quad\mid s$	
7.	$\quad\mid (p \rightarrow s)$	3-6 CP
8.	$((q \rightarrow s) \rightarrow (p \rightarrow s))$	2-7 CP

Nested conditional proofs

(107) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(108) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(109) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(110) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(111) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(112) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(113) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

What are the next steps?

Given the premise 1. we can prove $((q \rightarrow s) \rightarrow (p \rightarrow s))$!

(114) *conditional proof*:

$$\begin{array}{lll}
1. & (p \rightarrow (q \wedge r)) & \\
2. & \quad | \ (q \rightarrow s) & \text{Aux} \\
3. & \quad | \quad | \ p & \text{Aux} \\
4. & \quad | \quad | \ (q \wedge r) & \text{1,3 MP} \\
5. & \quad | \quad | \ q & \text{4 Simpl} \\
6. & \quad | \quad | \ s & \\
7. & \quad | \ (p \rightarrow s) & \text{3-6 CP} \\
8. & ((q \rightarrow s) \rightarrow (p \rightarrow s)) & \text{2-7 CP}
\end{array}$$

Nested conditional proofs

(107) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(108) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(109) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(110) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(111) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(112) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(113) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

What are the next steps?

Given the premise 1. we can prove $((q \rightarrow s) \rightarrow (p \rightarrow s))$!

(114) *conditional proof*:

1.	$(p \rightarrow (q \wedge r))$	
2.	$\mid (q \rightarrow s)$	Aux
3.	$\mid \mid p$	Aux
4.	$\mid \mid (q \wedge r)$	1,3 MP
5.	$\mid \mid q$	4 Simpl
6.	$\mid \mid s$	2,5 MP
7.	$\mid (p \rightarrow s)$	3-6 CP
8.	$((q \rightarrow s) \rightarrow (p \rightarrow s))$	2-7 CP

Direct conditional proofs: Exercise

(115) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(116) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(117) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(118) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(119) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(120) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(121) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

What is the auxiliary assumption?

Given the premises 1.-3. we can prove $(s \rightarrow (t \rightarrow u))$!

(122) *conditional proof*:

1.	$(p \vee (t \rightarrow u))$	
2.	$(p \rightarrow q)$	
3.	$((\neg s) \vee (\neg q))$	
4.		Aux
5.		
6.		
7.		
8.	$(s \rightarrow (t \rightarrow u))$	4-7 CP

Direct conditional proofs: Exercise

(115) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(116) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(117) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(118) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(119) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(120) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(121) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

What is the auxiliary assumption?
What are the next steps?

Given the premises 1.-3. we can prove $(s \rightarrow (t \rightarrow u))$!

(122) *conditional proof*:

1.	$(p \vee (t \rightarrow u))$	
2.	$(p \rightarrow q)$	
3.	$((\neg s) \vee (\neg q))$	
4.	$\quad \mid s$	Aux
5.	$\quad \mid$	
6.	$\quad \mid$	
7.	$\quad \mid (t \rightarrow u)$	
8.	$(s \rightarrow (t \rightarrow u))$	4-7 CP

Direct conditional proofs: Exercise

(115) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(116) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(117) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(118) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(119) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(120) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(121) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

What is the auxiliary assumption?
What are the next steps?

Given the premises 1.-3. we can prove $(s \rightarrow (t \rightarrow u))$!

(122) *conditional proof*:

1.	$(p \vee (t \rightarrow u))$	
2.	$(p \rightarrow q)$	
3.	$((\neg s) \vee (\neg q))$	
4.	$\quad\vert\ s$	Aux
5.	$\quad\vert\ (\neg q)$	3,4 DS
6.	$\quad\vert$	
7.	$\quad\vert\ (t \rightarrow u)$	
8.	$(s \rightarrow (t \rightarrow u))$	4-7 CP

Direct conditional proofs: Exercise

(115) **Modus Ponens**

$$\begin{array}{ll} & (p \to q) \\ & p \\ \hline \therefore & q \end{array}$$

(116) **Modus Tollens**

$$\begin{array}{ll} & (p \to q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(117) **Hyp. Syll.**

$$\begin{array}{ll} & (p \to q) \\ & (q \to r) \\ \hline \therefore & (p \to r) \end{array}$$

(118) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(119) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(120) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(121) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

What is the auxiliary assumption?
What are the next steps?

Given the premises 1.-3. we can prove $(s \to (t \to u))$!

(122) *conditional proof*:

1.	$(p \vee (t \to u))$	
2.	$(p \to q)$	
3.	$((\neg s) \vee (\neg q))$	
4.	$\quad\vert\ s$	Aux
5.	$\quad\vert\ (\neg q)$	3,4 DS
6.	$\quad\vert\ (\neg p)$	2,5 MT
7.	$\quad\vert\ (t \to u)$	
8.	$(s \to (t \to u))$	4-7 CP

Direct conditional proofs: Exercise

(115) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(116) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(117) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(118) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(119) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(120) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(121) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

What is the auxiliary assumption?
What are the next steps?

Given the premises 1.-3. we can prove $(s \rightarrow (t \rightarrow u))$!

(122) *conditional proof*:

1.	$(p \vee (t \rightarrow u))$	
2.	$(p \rightarrow q)$	
3.	$((\neg s) \vee (\neg q))$	
4.	$\mid\; s$	Aux
5.	$\mid\; (\neg q)$	3,4 DS
6.	$\mid\; (\neg p)$	2,5 MT
7.	$\mid\; (t \rightarrow u)$	1,6 DS
8.	$(s \rightarrow (t \rightarrow u))$	4-7 CP

Indirect conditional proofs

- the types of proofs we have seen are all direct proofs and direct conditional proofs
- that means the goal of the proof has been to prove q from a premise p, where we have started with p and ended with q
- indirect proofs aim at contradictions
- this form of argumentation uses the logic of **reductio ad absurdum**, which we have seen earlier
  - we still start with premise p, but we introduce the **negation of the conclusion**, i.e. $(\neg q)$ **as an auxiliary premise**
  - then we try to derive a contradiction
  - if we derive a contradiction, then we have indirectly shown that q does follow from p by showing that $(\neg q)$ is not compatible with p
  - if we don't derive a contradiction, then we have indirectly shown that the proof is not valid after all and that q does not follow from p
- an indirect proof is a type of conditional proof since it uses an auxiliary premise
- unlike in the conditional proofs seen earlier, the auxiliary premise here is not a part of the conclusion: rather, it is the negation of the conclusion
- also: the conclusion itself doesn't need to be in a conditional form, it could even be an atomic statement

Indirect conditional proofs

(123) **Modus Ponens**

$$\begin{array}{ll} & (p \to q) \\ & p \\ \hline \therefore & q \end{array}$$

(124) **Modus Tollens**

$$\begin{array}{ll} & (p \to q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(125) **Hyp. Syll.**

$$\begin{array}{ll} & (p \to q) \\ & (q \to r) \\ \hline \therefore & (p \to r) \end{array}$$

(126) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(127) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(128) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(129) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

What is the auxiliary assumption?

Given the premises 1.-3. we can prove p !

(130) *indirect proof*:

1.	$(p \vee q)$	
2.	$(q \to r)$	
3.	$(\neg r)$	
4.		Aux
5.		
6.		
7.		
8.	p	4-7 IP

Indirect conditional proofs

(123) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(124) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(125) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(126) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(127) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(128) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(129) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

What is the auxiliary assumption?
What are the next steps?

Given the premises 1.-3. we can prove p !

(130) *indirect proof*:

1.	$(p \vee q)$	
2.	$(q \rightarrow r)$	
3.	$(\neg r)$	
4.	$\mid (\neg p)$	Aux
5.	$\mid$	
6.	$\mid$	
7.	$\mid$	
8.	p	4-7 IP

Indirect conditional proofs

(123) **Modus Ponens**

$$\begin{array}{ll} & (p \to q) \\ & p \\ \hline \therefore & q \end{array}$$

(124) **Modus Tollens**

$$\begin{array}{ll} & (p \to q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(125) **Hyp. Syll.**

$$\begin{array}{ll} & (p \to q) \\ & (q \to r) \\ \hline \therefore & (p \to r) \end{array}$$

(126) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(127) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(128) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(129) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

What is the auxiliary assumption?
What are the next steps?

Given the premises 1.-3. we can prove p !

(130) *indirect proof*:

1.	$(p \vee q)$	
2.	$(q \to r)$	
3.	$(\neg r)$	
4.	$\mid$ $(\neg p)$	Aux
5.	$\mid$ q	1,4 DS
6.	$\mid$	
7.	$\mid$	
8.	p	4-7 IP

Indirect conditional proofs

(123) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(124) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

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What is the auxiliary assumption?
What are the next steps?

Given the premises 1.-3. we can prove p !

(130) *indirect proof*:

1.	$(p \vee q)$	
2.	$(q \rightarrow r)$	
3.	$(\neg r)$	
4.	│ $(\neg p)$	Aux
5.	│ q	1,4 DS
6.	│ r	2,5 MP
7.	│	
8.	p	4-7 IP

Indirect conditional proofs

(123) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

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$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

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$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(129) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

What is the auxiliary assumption?
What are the next steps?

Given the premises 1.-3. we can prove p !

(130) *indirect proof*:

$$\begin{array}{lll} 1. & (p \vee q) & \\ 2. & (q \rightarrow r) & \\ 3. & (\neg r) & \\ 4. & \quad\mid (\neg p) & \text{Aux} \\ 5. & \quad\mid q & \text{1,4 DS} \\ 6. & \quad\mid r & \text{2,5 MP} \\ 7. & \quad\mid (r \wedge (\neg r)) & \text{3,6 Conj} \\ 8. & p & \text{4-7 IP} \end{array}$$

Indirect conditional proofs: Exercise

(131) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(132) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(133) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(134) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

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$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(136) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(137) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

What is the auxiliary assumption?
What are the next steps?

Given the premises 1.-3. we can prove m!

(138) *indirect proof*:

1.	$(\neg m) \rightarrow (n \wedge o)$	
2.	$(n \rightarrow p)$	
3.	$(o \rightarrow (\neg p))$	
4.		Aux
5.		
6.		
7.		
8.		
9.		
10.		
11.	m	4-10 IP

Indirect conditional proofs

(139) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(140) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(141) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(142) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(143) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(144) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

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Indirect conditional proofs

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2.	$(n \rightarrow p)$	
3.	$(o \rightarrow (\neg p))$	
4.	$\mid (\neg m)$	Aux
5.	$\mid$	
6.	$\mid$	
7.	$\mid$	
8.	$\mid$	
9.	$\mid$	
10.	$\mid$	
11.	m	4-10 IP

Indirect conditional proofs

(139) **Modus Ponens**

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4.	$\mid (\neg m)$	Aux
5.	$\mid (n \wedge o)$	1,4 MP
6.	$\mid$	
7.	$\mid$	
8.	$\mid$	
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Indirect conditional proofs

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$$\begin{array}{lll} 1. & (\neg m) \rightarrow (n \wedge o) & \\ 2. & (n \rightarrow p) & \\ 3. & (o \rightarrow (\neg p)) & \\ 4. & \quad | \; (\neg m) & \text{Aux} \\ 5. & \quad | \; (n \wedge o) & \text{1,4 MP} \\ 6. & \quad | \; n & \text{5 Simpl} \\ 7. & \quad | & \\ 8. & \quad | & \\ 9. & \quad | & \\ 10. & \quad | & \\ 11. & m & \text{4-10 IP} \end{array}$$

Indirect conditional proofs

(139) **Modus Ponens**

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1.	$(\neg m) \rightarrow (n \wedge o)$	
2.	$(n \rightarrow p)$	
3.	$(o \rightarrow (\neg p))$	
4.	$\vert\ (\neg m)$	Aux
5.	$\vert\ (n \wedge o)$	1,4 MP
6.	$\vert\ n$	5 Simpl
7.	$\vert\ o$	5 Simpl
8.	$\vert$	
9.	$\vert$	
10.	$\vert$	
11.	m	4-10 IP

Indirect conditional proofs

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Given the premises 1.-3. we can prove m!

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2.	$(n \rightarrow p)$		
3.	$(o \rightarrow (\neg p))$		
4.		$(\neg m)$	Aux
5.		$(n \wedge o)$	1,4 MP
6.		n	5 Simpl
7.		o	5 Simpl
8.		p	2,6 MP
9.			
10.			
11.	m		4-10 IP

Indirect conditional proofs

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$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

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$$\begin{array}{lll} 1. & (\neg m) \rightarrow (n \wedge o) & \\ 2. & (n \rightarrow p) & \\ 3. & (o \rightarrow (\neg p)) & \\ 4. & \quad | \ (\neg m) & \text{Aux} \\ 5. & \quad | \ (n \wedge o) & \text{1,4 MP} \\ 6. & \quad | \ n & \text{5 Simpl} \\ 7. & \quad | \ o & \text{5 Simpl} \\ 8. & \quad | \ p & \text{2,6 MP} \\ 9. & \quad | \ (\neg p) & \text{3,7 MP} \\ 10. & \quad | & \\ 11. & m & \text{4-10 IP} \end{array}$$

Indirect conditional proofs

(139) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

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$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(141) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(142) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(143) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(144) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(145) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

What is the auxiliary assumption?
What are the next steps?

Given the premises 1.-3. we can prove m!

(146) *indirect proof*:

1.	$(\neg m) \rightarrow (n \wedge o)$	
2.	$(n \rightarrow p)$	
3.	$(o \rightarrow (\neg p))$	
4.	$\mid (\neg m)$	Aux
5.	$\mid (n \wedge o)$	1,4 MP
6.	$\mid n$	5 Simpl
7.	$\mid o$	5 Simpl
8.	$\mid p$	2,6 MP
9.	$\mid (\neg p)$	3,7 MP
10.	$\mid (p \wedge (\neg p))$	8,9 Conj
11.	m	4-10 IP