Formale Grundlagen (Logik)
Modul 04-006-1001

Statement Logic III

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“Mathematical Methods in Linguistics”)

Recap: Statement logic

- we will assume an infinite vocabulary of atomic statements

Statement logic

A formal system where the primitives are all statements.

(1) *Basic expressions of statement logic*

a $p, q, r, s, p', p'', \ldots$

(2) *Syntax of statement logic*

a. An atomic statement is a well-formed formula.

b. If ϕ is a well-formed formula, then $(\neg\phi)$ is a well-formed formula.

c. If ϕ and ψ are well-formed formulas, then $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, and $(\phi \leftrightarrow \psi)$ are well-formed formulas.

d. Nothing else is a formula.

Recap: Statement logic

- we wrote down the semantic rules like the syntactic rules
- this is an alternative to truth tables
- read $[\![\ ]\!]^M$ as interpreted in relation to model M

(3) *Semantics of statement logic*

a. If ϕ is a formula, then $[\![(\neg\phi)]\!]^M = 1$ iff $[\![\phi]\!]^M = 0$.

b. If ϕ and ψ are formulas, then $[\![(\phi \wedge \psi)]\!]^M = 1$ iff both $[\![\phi]\!]^M = 1$ and $[\![\psi]\!]^M = 1$.

c. If ϕ and ψ are formulas, then $[\![(\phi \vee \psi)]\!]^M = 1$ iff at least one of $[\![\phi]\!]^M, [\![\psi]\!]^M = 1$.

d. If ϕ and ψ are formulas, then $[\![(\phi \rightarrow \psi)]\!]^M = 1$ iff either $[\![\phi]\!]^M = 0$ or $[\![\psi]\!]^M = 1$.

e. If ϕ and ψ are formulas, then $[\![(\phi \leftrightarrow \psi)]\!]^M = 1$ iff $[\![\phi]\!]^M = [\![\psi]\!]^M$.

Recap: Tautologies, contradictions & contingencies

- a tautological statement is always true: the final column in its truth table contains only the values 1/True, regardless of what the truth values of its atomic statements are

(4)

p	$(p \rightarrow p)$
1	1
0	1

- a logically contradictory statement is always false: the final column of its truth table only contains the values 0/False, regardless of what the truth values of its atomic statements are

(5)

p	$(\neg p)$	$(p \wedge (\neg p))$
1	0	0
0	1	0

- all other statements, with both 1/True and 0/False in the final column of their truth table are called logical contingencies

Logical equivalence & logical consequence

- if a biconditional statement ($P \leftrightarrow Q$) is a logical tautology, then the two constituent statements on either side of the biconditional arrow are logically equivalent
- to denote logical equivalence between two arbitrary expressions P and Q we write $P \Leftrightarrow Q$
- if a conditional statement ($P \rightarrow Q$) is a logical tautology, we say that the consequent is a logical consequence of the antecedent
- alternatively, we say that the antecedent logically implies the consequent; in both cases, we write $P \Rightarrow Q$

Logical equivalence: exercise

- let us prove another logical equivalence!

(6) $(p \rightarrow q) \Leftrightarrow ((\neg p) \vee q)$

Logical equivalence: exercise

- let us prove another logical equivalence!

(7) $(p \to q) \Leftrightarrow ((\neg p) \vee q)$

(8)

p	q	$(p \to q)$
1	1	1
1	0	0
0	1	1
0	0	1

(9)

p	q	$(\neg p)$	$((\neg p) \vee q)$
1	1	0	1
1	0	0	0
0	1	1	1
0	0	1	1

Formal components of a proof

- we will turn to one of the central uses of statement logic: constructing a proof/argument
- it consists of two parts
  1. a number of statements, called **premises**: these are just statements that we, for the sake of argument, assume to be True
  2. a **conclusion**, whose truth is demonstrated to necessarily follow from the assumed truth of the premises

(10)

$$\begin{array}{rc} & \text{premise 1} \\ & \text{premise 2} \\ \hline \therefore & \text{conclusion} \end{array}$$

- a proof is called **valid** iff there is no uniform assignment of truth values to its atomic statements which makes all its premises true and its conclusion false
- a proof is called **invalid** iff there is at least one uniform assignment of truth values to its atomic statements which makes all its premises true and its conclusion false

Formal components of a proof

- premises and conclusion of a proof are related by the conditional $\rightarrow$ (antecedent $\rightarrow$ conclusion)
- the premises are the antecedent of the conditional
- the conclusion is the consequent of the conditional

(11) For a given proof X, if $p_1, p_2, \ldots, p_n$ are premises of X and q the conlusion of X, then:

a. X is valid iff: $((p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q)$ is a tautology (i.e. always true)

b. X is invalid iff: $((p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q)$ is not a tautology (i.e. not always true)

- an example for a simple natural language proof:

(12)

	If Marie eats another pizza, she will get sick.
	Marie eats another pizza.
$\therefore$	Marie gets sick.

A kind of proof: Modus Ponens

- this proof is called Modus Ponens

(13)
If Marie eats another pizza, she will get sick.
Marie eats another pizza.
$\therefore$ Marie gets sick.

- we can translate this argument into statement logic:

(14) a. $p =$ Marie eats another pizza.

b. $q =$ Marie gets sick.

- thus, we get the following:

(15)
$(p \rightarrow q)$
p
$\therefore$ q

A kind of proof: Modus Ponens

- Modus Ponens:

(16)
$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

- we can show that the proof is valid with a truth table

(17)

p	q	$(p \rightarrow q)$	$((p \rightarrow q) \wedge p)$	$(((p \rightarrow q) \wedge p) \rightarrow q)$
1	1	1	1	1
1	0	0	0	1
0	1	1	0	1
0	0	1	0	1

- premises are connected via $\wedge$, the conclusion is a logical consequence ($\Rightarrow$) if the proof is valid, i.e. if the implication ($\rightarrow$) is a tautology

More proofs: Modus Tollens

- the following proof is called **Modus Tollens**

(18) If Jack drinks beer, he will get drunk.
Jack doesn't get drunk.

∴ Jack doesn't drink beer.

(19) $(p \rightarrow q)$
$(\neg q)$

$\therefore$ $(\neg p)$

- again, we can show the validity of the proof by means of a truth table

(20)

p	q	$(p \rightarrow q)$	$((p \rightarrow q) \wedge (\neg q))$	$(((p \rightarrow q) \wedge (\neg q)) \rightarrow (\neg p))$
1	1	1	0	1
1	0	0	0	1
0	1	1	0	1
0	0	1	1	1

More proofs: Hypothetical Syllogism

- the following proof is called Hypothetical Syllogism

(21)

	If Jack drinks, he falls asleep.
	If Jack sleeps, Sue gets angry.
∴	If Jack drinks, Sue gets angry.

(22)

$$\begin{array}{rl} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

- convince yourself of the validity of the proof in the tutorials (or at home) by constructing a truth table!

More proofs: Disjunctive Syllogism

- the next proof is called Disjunctive Syllogism

(23)

	Jill will eat or sleep.
	Jill will not eat.
$\therefore$	Jill will sleep.

(24)

$$\begin{array}{ll} & (p \lor q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

- the validity of the proof is illustrated by the following truth table

(25)

p	q	$(\neg p)$	$(p \lor q)$	$((p \lor q) \land (\neg p))$	$(((p \lor q) \land (\neg p)) \to q)$
1	1	0	1	0	1
1	0	0	1	0	1
0	1	1	1	1	1
0	0	1	0	0	1

More proofs: Simplification

- the next proof is called Simplification

(26) Bill is short and Marie is tall.
$\therefore$ Bill is short.

(27)
$$\frac{(p \wedge q)}{\therefore \quad p}$$

- show the validity of the proof with a truth table (solution on next page)

More proofs: Simplification

- here is the truth table that shows the validity of the proof for simplification:

(28)

p	q	$(p \wedge q)$	$((p \wedge q) \rightarrow p)$
1	1	1	1
1	0	0	1
0	1	0	1
0	0	0	1

More proofs: Conjunction

- here is a proof called Conjunction

(29)

	Bill is short.
	Marie is tall.
∴	Bill is short and Marie is tall.

(30)

	p
	q
∴	$(p \wedge q)$

- the validity of the proof by means of a truth table is as follows

(31)

p	q	$(p \wedge q)$	$((p \wedge q) \rightarrow (p \wedge q))$
1	1	1	1
1	0	1	1
0	1	1	1
0	0	0	1

More proofs: Addition

- the next proof is called Addition

(32) Bill is short.
$\therefore$ Bill is short or the earth is round.

(33)
$$\frac{p}{\therefore\ (p \vee q)}$$

- show the validity of the proof with a truth table (solution on next page)

More proofs: Addition

- here is the truth table that shows the validity of the proof for addition:

(34)

p	q	$(p \vee q)$	$(p \rightarrow (p \vee q))$
1	1	1	1
1	0	1	1
0	1	1	1
0	0	0	1

Logical Fallacies

- the proof below is an invalid argument!
- this particular logical fallacy is called: **fallacy of affirming the consequent**

(35)
$$\begin{array}{ll} & (p \rightarrow q) \\ & q \\ \hline \not\therefore & p \end{array}$$

- we can show that the proof is invalid with a truth table
- construct the truth table for this invalid proof. what would we expect as truth values in the last column (solution on next page)?

Logical Fallacies

- an invalid proof is defined as a conditional that does not always result in True
- this is illustrated by the following truth table for the fallacy of affirming the consequent

(36)

p	q	$(p \to q)$	$((p \to q) \land q)$	$(((p \to q) \land q) \to p)$
1	1	1	1	1
1	0	0	0	1
0	1	1	1	0
0	0	1	0	1

Logical Fallacies

- fallacy of affirming the consequent:

(37)
$$\begin{array}{ll} & (p \rightarrow q) \\ & q \\ \hline \therefore\!\!\!/ & p \end{array}$$

- it is easy to see why the proof is invalid: the truth of q does not necessarily entail/imply the truth of the conclusion
- take the following natural language equivalent!

(38)
	If Marie eats another pizza, she will get sick.
	Marie gets sick.
$\therefore\!\!\!/$	Marie eats another pizza.

- Marie could have gotten sick for a million different reasons

Logical Fallacies

- here is another invalid argument
- this particular logical fallacy is called: **fallacy of denying the antecedent**

(39)
$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg p) \\ \hline \therefore\!\!\!/ & (\neg q) \end{array}$$

- show that the proof is invalid with a truth table (solution on next page)
- recall that an invalid proof is defined as a conditional that does not always result in True

Logical Fallacies

- the truth table for showing the fallacy of denying the antecedent:

(40)

p	q	$(\neg p)$	$(\neg q)$	$(p \rightarrow q)$	$((p \rightarrow q) \wedge (\neg p))$	$(((p \rightarrow q) \wedge (\neg p)) \rightarrow (\neg q))$
1	1	0	0	1	0	1
1	0	0	1	0	0	1
0	1	1	0	1	1	0
0	0	1	1	1	1	1

Logical Fallacies

- fallacy of denying the antecedent:

(41)

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg p) \\ \hline \therefore\!\!\!/ & (\neg q) \end{array}$$

- again, it is easy to see why the proof is invalid: denying the truth of p does not necessarily entail/imply the falsity of the conclusion
- let us think of a natural language equivalent!

(42)

$$\begin{array}{ll} & \text{If Marie eats another pizza, she will get sick.} \\ & \text{Marie doesn't eat another pizza.} \\ \hline \therefore\!\!\!/ & \text{Marie will not get sick.} \end{array}$$

- Marie can get sick nevertheless, for different reasons, e.g. too many cocktails

Simple proofs

(43) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(44) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(45) **Hypothetical Syllogism**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(46) **Disjunctive Syllogism**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(47) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(48) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(49) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

Given the premises 1.-5. we can prove the atomic statement t !

(50) *simple proof*:

1.	$(p \rightarrow q)$	
2.	$(p \vee s)$	
3.	$(q \rightarrow r)$	
4.	$(s \rightarrow t)$	
5.	$(\neg r)$	
6.	$(\neg q)$	3,5 MT
7.	$(\neg p)$	1,6 MT
8.	s	2,7 DS
9.	t	4,8 MP

Complex proofs

(51) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(52) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(53) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(54) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(55) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(56) **Identity Laws**:

a. $x \vee False \Leftrightarrow x$

b. $x \wedge False \Leftrightarrow False$

c. $x \vee True \Leftrightarrow True$

d. $x \wedge True \Leftrightarrow x$

(57) **Conditional Laws**:

a. $(p \rightarrow q) \Leftrightarrow ((\neg p) \vee q)$

b. $(p \rightarrow q) \Leftrightarrow ((\neg q) \rightarrow (\neg p))$

(58) **Commutative Laws**:

a. $(p \vee q) \Leftrightarrow (q \vee p)$

b. $(p \wedge q) \Leftrightarrow (q \wedge p)$

(59) **Associative Laws**:

a. $((p \vee q) \vee r) \Leftrightarrow (p \vee (q \vee r))$

b. $((p \wedge q) \wedge r) \Leftrightarrow (p \wedge (q \wedge r))$

Given the premises 1.-2. we can prove the implication $(p \rightarrow q)$!

(60) *complex proof*:

1. $(p \rightarrow (q \vee r))$
2. $(\neg r)$
3. $((\neg p) \vee (q \vee r))$ 1 Cond
4. $(((\neg p) \vee q) \vee r)$ 3 Ass
5. $(((\neg p) \vee q) \vee F)$ 4 Neg
6. $(((\neg p) \vee q))$ 5 Ident
7. $(p \rightarrow q)$ 6 Cond