Formale Grundlagen (Logik)
Modul 04-006-1001

Statement Logic II

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(Slides by Imke Driemel & Sandhya Sundaresan,
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Recap: Orderings, cardinality

- an order is a binary relation which is **transitive** and additionally:

weak order	strong order
• reflexive	• irreflexive
• anti-symmetric	• asymmetric

- if an order (weak or strong) is also connected (i.e. every distinct element in A is related to another in an ordered pair) then it is a **total or linear** order
- the cardinality of a set = the number of members/elements inside that set

(1) a. $X = \{a, b, c\}$

b. $|X| = 3$

- if sets X and Y are equivalent (one-to-one correspondence), then they are also of the same size; common notation: $X \sim Y$
- equal vs. equivalence: two sets are equal iff they have the same members; set equivalence has to do with the number of members

Recap: Formal systems

- a formal system consists of:
  1. a non-empty set of primitives: the things/objects we are interested in investigating further
  2. a set of statements, called axioms, about those primitives
  3. a way to reason, i.e. make further statements from these axioms
- we have an intuitive understanding of which reasoning is valid and which is not
- if we accept the truth of the premise of a valid argument, we cannot deny its consequence
- within formal languages we separate between form and content
  - **syntax**:
    properties of expressions of the system itself, such as its primitives, axioms, rules of inference
  - **semantics**:
    relations between the system and its models or interpretations

Recap: Statement logic

- we will assume an infinite vocabulary of atomic statements

Statement logic

A formal system where the primitives are all statements.

(2) *Basic expressions of statement logic*

$p, q, r, s, p', p'', \ldots$

(3) *Syntax of statement logic*

a. An atomic statement is a well-formed formula.

b. If ϕ is a well-formed formula, then $(\neg\phi)$ is a well-formed formula.

c. If ϕ and ψ are well-formed formulas, then $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, and $(\phi \leftrightarrow \psi)$ are well-formed formulas.

d. Nothing else is a formula.

Recap: Statement logic

- we can write down the semantic rules like the syntactic rules
- read $[\![\ ]\!]^M$ as interpreted in relation to model M

(4) *Semantics of statement logic*

a. If ϕ is a formula, then $[\![(\neg\phi)]\!]^M = 1$ iff $[\![\phi]\!]^M = 0$.

b. If ϕ and ψ are formulas, then $[\![(\phi \wedge \psi)]\!]^M = 1$ iff both $[\![\phi]\!]^M = 1$ and $[\![\psi]\!]^M = 1$.

c. If ϕ and ψ are formulas, then $[\![(\phi \vee \psi)]\!]^M = 1$ iff at least one of $[\![\phi]\!]^M$, $[\![\psi]\!]^M = 1$.

d. If ϕ and ψ are formulas, then $[\![(\phi \rightarrow \psi)]\!]^M = 1$ iff $[\![\phi]\!]^M = 0$ or $[\![\psi]\!]^M = 1$.

e. If ϕ and ψ are formulas, then $[\![(\phi \leftrightarrow \psi)]\!]^M = 1$ iff $[\![\phi]\!]^M = [\![\psi]\!]^M$.

(5) Truth table for negation:

p	$(\neg p)$
1	0
0	1

Recap: Statement logic

- we can write down the semantic rules like the syntactic rules
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(6) *Semantics of statement logic*

a. If ϕ is a formula, then $[\![(\neg\phi)]\!]^M = 1$ iff $[\![\phi]\!]^M = 0$.

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e. If ϕ and ψ are formulas, then $[\![(\phi \leftrightarrow \psi)]\!]^M = 1$ iff $[\![\phi]\!]^M = [\![\psi]\!]^M$.

(7) Truth table for conjunction:

p	q	$(p \wedge q)$
1	1	1
1	0	0
0	1	0
0	0	0

Recap: Statement logic

- we can write down the semantic rules like the syntactic rules
- read $[\![\ ]\!]^M$ as interpreted in relation to model M

(8) *Semantics of statement logic*

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(9) Truth table for disjunction:

p	q	$(p \vee q)$
1	1	1
1	0	1
0	1	1
0	0	0

Recap: Statement logic

- we can write down the semantic rules like the syntactic rules
- read $[\![\ ]\!]^M$ as interpreted in relation to model M

(10) *Semantics of statement logic*

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b. If ϕ and ψ are formulas, then $[\![(\phi \wedge \psi)]\!]^M = 1$ iff both $[\![\phi]\!]^M = 1$ and $[\![\psi]\!]^M = 1$.

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e. If ϕ and ψ are formulas, then $[\![(\phi \leftrightarrow \psi)]\!]^M = 1$ iff $[\![\phi]\!]^M = [\![\psi]\!]^M$.

(11) Truth table for conditional:

p	q	$(p \rightarrow q)$
1	1	1
1	0	0
0	1	1
0	0	1

Recap: Statement logic

- we can write down the semantic rules like the syntactic rules
- read $[\![\ ]\!]^M$ as interpreted in relation to model M

(12) *Semantics of statement logic*

a. If ϕ is a formula, then $[\![(\neg\phi)]\!]^M = 1$ iff $[\![\phi]\!]^M = 0$.

b. If ϕ and ψ are formulas, then $[\![(\phi \wedge \psi)]\!]^M = 1$ iff both $[\![\phi]\!]^M = 1$ and $[\![\psi]\!]^M = 1$.

c. If ϕ and ψ are formulas, then $[\![(\phi \vee \psi)]\!]^M = 1$ iff at least one of $[\![\phi]\!]^M, [\![\psi]\!]^M = 1$.

d. If ϕ and ψ are formulas, then $[\![(\phi \rightarrow \psi)]\!]^M = 1$ iff $[\![\phi]\!]^M = 0$ or $[\![\psi]\!]^M = 1$.

e. If ϕ and ψ are formulas, then $[\![(\phi \leftrightarrow \psi)]\!]^M = 1$ iff $[\![\phi]\!]^M = [\![\psi]\!]^M$.

(13) Truth table for biconditional:

p	q	$(p \leftrightarrow q)$
1	1	1
1	0	0
0	1	0
0	0	1

Computing complex truth values

- so far, we've looked at truth tables for expressions that consist of maximally two atomic statements
- but a truth table provides a systematic method to compute the truth value of any expression in statement logic
- the number of rows in a truth table depends on the number of atomic statements: every logical combination has to show up
- if an expression has n atomic statements, then the truth table will have 2^n rows
- why 2? . . . because we have 2 truth values
- so let us compute the truth values of (depending on the truth values of the atomic statements p, q, and r):

(14) $((p \wedge q) \rightarrow (\neg(p \vee r))$

Computing complex truth values

- so let us compute the truth values of:

(15) $((p \wedge q) \rightarrow (\neg(p \vee r))$

(16)

p	q	r	$(p \wedge q)$	$(p \vee r)$	$\neg(p \vee r)$	$((p \wedge q) \rightarrow (\neg(p \vee r))$
1	1	1	1	1	0	0
1	1	0	1	1	0	0
1	0	1	0	1	0	1
1	0	0	0	1	0	1
0	1	1	0	1	0	1
0	1	0	0	0	1	1
0	0	1	0	1	0	1
0	0	0	0	0	1	1

Exercise

- last week, we said that the following holds:

(17) $(p \leftrightarrow q) = ((p \rightarrow q) \wedge (q \rightarrow p))$

- prove it!

Exercise

- last week, we said that the following holds:

(18) $(p \leftrightarrow q) = ((p \rightarrow q) \wedge (q \rightarrow p))$

- prove it!

(19)

p	q	$(p \leftrightarrow q)$
1	1	1
1	0	0
0	1	0
0	0	1

(20)

p	q	$(p \rightarrow q)$	$(q \rightarrow p)$	$((p \rightarrow q) \wedge (q \rightarrow p))$
1	1	1	1	1
1	0	0	1	0
0	1	1	0	0
0	0	1	1	1

Tautologies

- a (complex) statement is called a logical tautology iff the final column in its truth table contains only the values 1/True, regardless of what the truth values of its atomic statements are
- a tautological statement is true simply because of the meaning of the logical connective(s) in it: this is why the meanings of the individual atomic statements in it don't matter
- another way to express this would be to say that a tautological statement is always true

(21) example of a logical tautology:

$(p \rightarrow p)$

p	$(p \rightarrow p)$
1	1
0	1

Contradictions

- a (complex) statement is called a logical contradiction iff the final column of its truth table only contains the values 0/False, regardless of what the truth values of its atomic statements are
- similarly to a tautology, a contradictory statement is false simply because of the meaning of the logical connective(s) in it: this is why the meanings of the individual atomic statements in it don't matter
- a logically contradictory statement is always false

(22) example of a logical contradiction:

$(p \wedge (\neg p))$

p	$(\neg p)$	$(p \wedge (\neg p))$
1	0	0
0	1	0

Contingencies

- an important property of logical tautologies and logical contradictions is that the truth values of the atomic statements in them simply don't matter
- all overall truth values (final column) are either always 1 (tautologies) or always 0 (contradictions)
- all other statements, with both 1/True and 0/False in the final column of their truth table are called logical contingencies
- the idea is that the truth of these statements is contingent (i.e. dependent) on the truth of the atomic statement(s) contained in them
- most of the examples (involving both complex and atomic statements) we've seen so far have been logical contingencies

(23) example of a contingency:

$((p \vee q) \rightarrow q)$

p	q	$(p \vee q)$	$((p \vee q) \rightarrow q)$
1	1	1	1
1	0	1	0
0	1	1	1
0	0	0	1

Exercise

- among the three complex expressions below, one is a logical tautology, one is a logical contradiction, and one is a logical contingency

(24) a. $(p \rightarrow (q \rightarrow p))$

b. $(p \vee q)$

c. $(\neg(p \vee (\neg p)))$

- say which is which by drawing truth tables for each of the expressions

Exercise

- among the three complex expressions below, one is a logical tautology, one is a logical contradiction, and one is a logical contingency

(24) a. $(p \rightarrow (q \rightarrow p))$
b. $(p \vee q)$
c. $(\neg(p \vee (\neg p)))$

- say which is which by drawing truth tables for each of the expressions
- $(p \rightarrow (q \rightarrow p))$ is a tautology!

(25)

p	q	$(q \rightarrow p)$	$(p \rightarrow (q \rightarrow p))$
1	1	1	1
1	0	1	1
0	1	0	1
0	0	1	1

Exercise

- among the three complex expressions below, one is a logical tautology, one is a logical contradiction, and one is a logical contingency

(24) a. $(p \rightarrow (q \rightarrow p))$

b. $(p \vee p)$

c. $(\neg(p \vee (\neg p)))$

- say which is which by drawing truth tables for each of the expressions
- $(p \vee q)$ is a contingency!

(26)

p	p	$(p \vee p)$
1	1	1
0	0	0

Exercise

- among the three complex expressions below, one is a logical tautology, one is a logical contradiction, and one is a logical contingency

(24) a. $(p \rightarrow (q \rightarrow p))$

b. $(p \vee p)$

c. $(\neg(p \vee (\neg p)))$

- say which is which by drawing truth tables for each of the expressions
- $(\neg(p \vee (\neg p)))$ is a contradiction!

(27)

p	$\neg p$	$(p \vee (\neg p))$	$(\neg(p \vee (\neg p)))$
1	0	1	0
0	1	1	0

reductio ad absurdum

- another way of reasoning (one without truth tables), i.e., proving that some statement is a tautology, is called reductio ad absurdum
- the reasoning works like this (we have done this before):
  1. assume that the statement is in fact not a logical tautology, i.e. you assume that one of its possible truth values is 0
  2. then reason “backwards” from this assumption to compute the possible values of the atomic statements in this complex expression
  3. if, based on this assumption, you run into a contradiction, then your assumption was wrong and the statement is indeed a tautology
  4. if, on the other hand, you don’t run into a contradiction, then your assumption was correct after all, and the complex statement is not a logical tautology

reductio ad absurdum: example

- here is an example: suppose we want to prove that $(p \rightarrow (q \rightarrow p))$ is a tautology (see above)
  1. assume that $(p \rightarrow (q \rightarrow p))$ is false:

     (28) $$\begin{array}{ccccc} (p & \rightarrow & (q & \rightarrow & p)) \\ & 0 & & & \end{array}$$

  2. reasoning backwards from this assumption, we know that the only way the whole expression can be 0 is if the antecedent is 1 and the consequent is 0

     (29) $$\begin{array}{ccccc} (p & \rightarrow & (q & \rightarrow & p)) \\ 1 & 0 & & 0 & \end{array}$$

  3. now we have to give every instance of p the same value (p is a constant)

     (30) $$\begin{array}{ccccc} (p & \rightarrow & (q & \rightarrow & p)) \\ 1 & 0 & & 0 & 1 \end{array}$$

  4. contradiction: there is no way a conditional ($(q \rightarrow p)$) can be false if the consequent (p) is true (this holds independent of the truth value of q)
  5. since our assumption that $(p \rightarrow (q \rightarrow p))$ is false has led us to a contradiction, our assumption must be false, hence $(p \rightarrow (q \rightarrow p))$ is a logical tautology

reductio ad absurdum: exercise

- try this line of reasoning with the next example

(31) $(p \vee (\neg p))$

1. assume that $(p \vee (\neg p))$ is false:

   (32) $\begin{array}{ccccc} (p & \vee & (\neg & p)) \\ & 0 & & \end{array}$

2. reasoning backwards from this assumption, we know that the only way the whole expression can be 0 is if both disjuncts are 0

   (33) $\begin{array}{ccccc} (p & \vee & (\neg & p)) \\ 0 & 0 & 0 & \end{array}$

3. a contradiction: there is no way p can be 0 and $\neg p$ can be 0
4. since our assumption that $(p \vee (\neg p))$ is false has led us to a contradiction, our assumption must be false, hence $(p \vee (\neg p))$ is a logical tautology

Logical equivalence

- if a biconditional statement is a logical tautology, then the two constituent statements on either side of the biconditional arrow are logically equivalent
- in other words: a biconditional between p and q is True precisely if they both are True or if they both are False, hence p and q always need to give back the same truth value for logical equivalence

(34)

p	q	$(p \leftrightarrow q)$
1	1	1
1	0	0
0	1	0
0	0	1

- to denote logical equivalence between two arbitrary expressions P and Q (atomic or complex) we write $P \Leftrightarrow Q$
- we have already proven a logical equivalence:

(35) $(p \leftrightarrow q) \Leftrightarrow ((p \rightarrow q) \wedge (q \rightarrow p))$

Logical equivalence: exercise

- let us prove another logical equivalence!

(36) $(\neg(p \vee q)) \Leftrightarrow ((\neg p) \wedge (\neg q))$

Logical equivalence: exercise

- let us prove another logical equivalence!

(36) $(\neg(p \vee q)) \Leftrightarrow ((\neg p) \wedge (\neg q))$

(37)

p	q	$(p \vee q)$	$(\neg(p \vee q))$
1	1	1	0
1	0	1	0
0	1	1	0
0	0	0	1

(38)

p	q	$(\neg p)$	$(\neg q)$	$((\neg p) \wedge (\neg q))$
1	1	0	0	0
1	0	0	1	0
0	1	1	0	0
0	0	1	1	1

Logical equivalence

- regardless of the truth values of the atomic statements p and q, the expressions $(\neg(p \vee q))$ and $((\neg p) \wedge (\neg q))$ always have the same truth value
- since two logically equivalent statements have exactly the same truth values in every row of the truth table, one can substitute one for the other in a larger expression E, and vice versa, without changing the truth value of E
- so we can subtitute $((\neg p) \wedge (\neg q))$ with $(\neg(p \vee q))$ (and vice versa)
- another example:

(39) $p \Leftrightarrow (p \wedge p)$

  a. $((p \wedge p) \vee q)$

  b. substitution: $(p \vee q)$

- the following laws of statement logic define various logic equivalences

Laws of Statement logic

(40) **Idempotent Laws**:

a. $(P \vee P) \Leftrightarrow P$

b. $(P \wedge P) \Leftrightarrow P$

(41) **Commutative Laws**:

a. $(P \vee Q) \Leftrightarrow (Q \vee P)$

b. $(P \wedge Q) \Leftrightarrow (Q \wedge P)$

(42) **Associative Laws**:

a. $((P \vee Q) \vee R) \Leftrightarrow (P \vee (Q \vee R))$

b. $((P \wedge Q) \wedge R) \Leftrightarrow (P \wedge (Q \wedge R))$

(43) **Identity Laws**:

a. $(P \vee \mathit{False}) \Leftrightarrow P$

b. $(P \wedge \mathit{False}) \Leftrightarrow \mathit{False}$

c. $(P \vee \mathit{True}) \Leftrightarrow \mathit{True}$

d. $(P \wedge \mathit{True}) \Leftrightarrow P$

(44) **Distributive Laws**:

a. $(P \vee (Q \wedge R)) \Leftrightarrow ((P \vee Q) \wedge (P \vee R))$

b. $(P \wedge (Q \vee R)) \Leftrightarrow ((P \wedge Q) \vee (P \wedge R))$

(45) **Complement Laws**:

a. $(P \vee (\neg P)) \Leftrightarrow \mathit{True}$

b. $(\neg(\neg P)) \Leftrightarrow P$

c. $(P \wedge (\neg P)) \Leftrightarrow \mathit{False}$

(46) **DeMorgan's Laws**:

a. $(\neg(P \vee Q)) \Leftrightarrow ((\neg P) \wedge (\neg Q))$

b. $(\neg(P \wedge Q)) \Leftrightarrow ((\neg P) \vee (\neg Q))$

Laws of Statement logic

(47) **Conditional Laws**:

a. $(P \rightarrow Q) \Leftrightarrow ((\neg P) \vee Q)$

b. $(P \rightarrow Q) \Leftrightarrow ((\neg Q) \rightarrow (\neg P))$

(48) **Biconditional Laws**:

a. $(P \leftrightarrow Q) \Leftrightarrow ((P \rightarrow Q) \wedge (Q \rightarrow P))$

b. $(P \leftrightarrow Q) \Leftrightarrow (((\neg P) \wedge (\neg Q)) \vee (P \wedge Q))$

Logical consequence

- if a conditional statement is a logical tautology, we say that the consequent is a logical consequence of the antecedent (*antecedent* $\rightarrow$ *consequent*)
- alternatively, we say that the antecedent logically implies the consequent, and we write this as $P \Rightarrow Q$

(49) example of a logical consequence:

$(((p \rightarrow q) \wedge q) \rightarrow q)$

p	q	$(p \rightarrow q)$	$((p \rightarrow q) \wedge q)$	$(((p \rightarrow q) \wedge q) \rightarrow q)$
1	1	1	1	1
1	0	0	0	1
0	1	1	1	1
0	0	1	0	1

- the rightmost column shows the conditional statement of which we want to find out if its consequent q is a logical consequence of the antecedent $((p \rightarrow q) \wedge q)$
- since every value in this column is True, the conditional is a tautology, and hence q is also a logical consequence of $((p \rightarrow q) \wedge q)$
- so we write: $((p \rightarrow q) \wedge q) \Rightarrow q$