Formale Grundlagen (Logik)
Modul 04-006-1001

Orderings, Introduction to Statement Logic

Leipzig University

December 5^{th}, 2024

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(Slides by Imke Driemel & Sandhya Sundaresan,
based on Partee, ter Meulen und Wall 1990
"Mathematical Methods in Linguistics")

Recap: Relations Reflexivity and Symmetry

- given a set A and a relation R in A ($R \subseteq A \times A$), R is **reflexive** iff all the ordered pairs of the form $\langle x, x \rangle$ are in R for every x in A
  - a relation that is not reflexive is called **non-reflexive**
  - a relation which contains no ordered pair of the form $\langle x, x \rangle$ is **irreflexive**
- "taller than" = irreflexive, "equal to" = reflexive, "is financial supporter of" = non-reflexive
- given a set A and a relation R in A ($R \subseteq A \times A$), R is **symmetric** iff for every ordered pair $\langle x, y \rangle$ in R, the pair $\langle y, x \rangle$ is also in R
  - a relation that is not symmetric is called **non-symmetric**
  - a relation in which it is never the case that for an ordered pair $\langle x, y \rangle$, $\langle y, x \rangle$ is also a member, is **asymmetric**
  - a relation is **anti-symmetric** if whenever both $\langle x, y \rangle$ and $\langle y, x \rangle$ are in R, then $x = y$
- "self-employed" = symmetric, anti-symmetric, "friend of" = non-symmetric, "father of" = asymmetric, and "cousin of" = symmetric

Recap: Relations Transitivity and Connectedness

- given a set A and a relation R in A ($R \subseteq A \times A$), R is **transitive** iff for all ordered pair $\langle x, y \rangle$ and $\langle y, z \rangle$ in R, the pair $\langle x, z \rangle$ is also in R
    - a relation that is not transitive is called **non-transitive**
    - a relation is **intransitive** if for no pairs $\langle x, y \rangle$ and $\langle y, z \rangle$ in R, the ordered pair $\langle x, z \rangle$ is in R
- "mother of" = intransitive, "older than" = transitive, "like" = non-transitive

- given a set A and a relation R in A ($R \subseteq A \times A$), R is **connected** or connex iff for every two distinct elements x and y in A, the pair $\langle x, y \rangle \in R$ or $\langle y, x \rangle \in R$ or both
- "father of" = not connected, "greater than" = connected, "same hair color as" = not connected

Recap: Properties of R^{-1} and R'

- recall that the inverse of a relation R (= R^{-1}) is simply R with the members inside each ordered pair reversed
- and that the complement of a relation R (= R') contains all the ordered pairs (in the Cartesian Product of which R is a subset) that are not in R
- certain properties are preserved from R to R^{-1} and R'

R (not $\varnothing$)	R^{-1}	R'
reflexive	reflexive	irreflexive
irreflexive	irreflexive	reflexive
symmetric	symmetric	symmetric
asymmetric	asymmetric	non-symmetric
antisymmetric	antisymmetric	depends on R
transitive	transitive	depends on R
intransitive	intransitive	depends on R
connected	connected	depends on R

Recap: Equivalence relations and classes; Partitions

- an **equivalence relation** is one which is **reflexive**, **symmetric**, and **transitive**
- "=" is the most typical equivalence relation; others are: "same height as" or "same age as "
- an equivalence class $[x]$ is a set of all elements that are related to x by some equivalence relation

(1) $[x] = \{y \mid \langle x, y \rangle \in R\}$, where R is an equivalence relation

- there is a close correspondence between equivalence classes and partitions
- given a non-empty set A, a **partition** of A is a collection of non-empty subsets of A (where subsets are called cells) such that
  1. for any two distinct subsets X and Y, $X \cap Y = \varnothing$
  2. the union of all the subsets equals A
- equivalence classes specified by R in set A are the cells of the partition induced on A!

Orderings

- an ordering is a binary relation which is **transitive** and additionally:

weak ordering	strong ordering
• reflexive	• irreflexive
• anti-symmetric	• asymmetric

- which of the following relations on set A are orderings? if so, are they strong or weak orderings?

(2) $A = \{a, b, c, d\}$

(3) a. $R_1 = \{\langle a,b\rangle, \langle a,c\rangle, \langle a,d\rangle, \langle b,c\rangle, \langle a,a\rangle, \langle b,b\rangle, \langle c,c\rangle, \langle d,d\rangle\}$ *weak*

b. $R_2 = \{\langle b,a\rangle, \langle c,b\rangle, \langle c,a\rangle\}$ *strong*

c. $R_3 = \{\langle a,b\rangle, \langle a,d\rangle, \langle b,c\rangle, \langle a,a\rangle, \langle b,b\rangle, \langle c,c\rangle, \langle d,d\rangle\}$ *not an ordering*

d. $R_4 = \{\langle d,c\rangle, \langle d,b\rangle, \langle d,a\rangle, \langle c,b\rangle, \langle c,a\rangle, \langle a,a\rangle, \langle b,b\rangle, \langle c,c\rangle, \langle d,d\rangle, \langle b,a\rangle\}$ *weak*

e. $R_5 = \{\langle a,b\rangle, \langle a,c\rangle, \langle a,d\rangle, \langle b,c\rangle\}$ *strong*

f. $R_6 = \{\langle b,a\rangle, \langle b,b\rangle, \langle a,a\rangle, \langle c,c\rangle, \langle d,d\rangle, \langle c,b\rangle, \langle c,a\rangle\}$ *weak*

g. $R_7 = \{\langle d,c\rangle, \langle d,b\rangle, \langle d,a\rangle, \langle c,b\rangle, \langle c,a\rangle, \langle b,a\rangle\}$ *strong*

h. $R_8 = \{\langle a,b\rangle, \langle a,c\rangle, \langle a,d\rangle, \langle b,c\rangle, \langle d,a\rangle\}$ *not an ordering*

Orderings

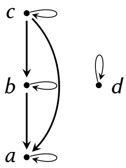
- an ordering is a binary relation which is **transitive** and additionally:

weak ordering

- reflexive
- anti-symmetric

(4) $R_6 = \{\langle b, a\rangle, \langle b, b\rangle, \langle a, a\rangle, \langle c, c\rangle, \langle d, d\rangle, \langle c, b\rangle, \langle c, a\rangle\}$

(5)

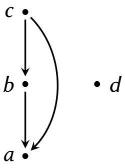


strong ordering

- irreflexive
- asymmetric

(6) $R_2 = \{\langle b, a\rangle, \langle c, b\rangle, \langle c, a\rangle\}$

(7)

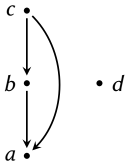


- we can produce a strong ordering from a weak ordering simply by removing all ordered pairs of the form $\langle x, x\rangle$
- conversely, we can convert a strong ordering into a weak ordering by simply adding all ordered pairs of the form $\langle x, x\rangle$ for every x in A

Orderings

- we can determine precedence relations on orderings
- If R is an ordering (weak or strong), and $\langle x, y \rangle \in R$, then:
  - x precedes y or x is a predecessor of y, and
  - y succeeds x, or y is a successor of x
- if x precedes y, $x \neq y$, and there is no element z distinct from both x and y such that x precedes z and z precedes y, then:
  - x immediately precedes y or x is an immediate predecessor of y, and
  - y immediately succeeds x, or y is an immediate successor of x
- in R_2 for example, c precedes a but not immediately since b intervenes; only b immediately precedes a

(8) a. $R_2 = \{\langle b, a \rangle, \langle c, b \rangle, \langle c, a \rangle\}$

b. 

Orderings

- if an ordering (weak or strong) is also connected (i.e. every distinct element in *A* is related to another in an ordered pair) then it is a **total or linear** ordering

- which of our orderings on set *A* are total/linear?

(9) $A = \{a, b, c, d\}$

(10) a. $R_1 = \{\langle a, b\rangle, \langle a, c\rangle, \langle a, d\rangle, \langle b, c\rangle, \langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle d, d\rangle\}$ *weak*

b. $R_2 = \{\langle b, a\rangle, \langle c, b\rangle, \langle c, a\rangle\}$ *strong*

c. $R_4 = \{\langle d, c\rangle, \langle d, b\rangle, \langle d, a\rangle, \langle c, b\rangle, \langle c, a\rangle, \langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle d, d\rangle, \langle b, a\rangle\}$ *weak*

d. $R_5 = \{\langle a, b\rangle, \langle a, c\rangle, \langle a, d\rangle, \langle b, c\rangle\}$ *strong*

e. $R_6 = \{\langle b, a\rangle, \langle b, b\rangle, \langle a, a\rangle, \langle c, c\rangle, \langle d, d\rangle, \langle c, b\rangle, \langle c, a\rangle\}$ *weak*

f. $R_7 = \{\langle d, c\rangle, \langle d, b\rangle, \langle d, a\rangle, \langle c, b\rangle, \langle c, a\rangle, \langle b, a\rangle\}$ *strong*

Orderings

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f. $R_7 = \{\langle d, c\rangle, \langle d, b\rangle, \langle d, a\rangle, \langle c, b\rangle, \langle c, a\rangle, \langle b, a\rangle\}$ *strong*

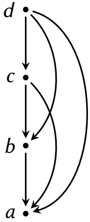
- only R_7 and R_4 are linear orderings

Orderings

- if an ordering (weak or strong) is also connected (i.e. every distinct element in A is related to another in an ordered pair) then it is a **total or linear** ordering
- we exemplify here with our strong orderings: R_7 is linear, R_2 and R_5 are not

(11) $R_7 = \{\langle d, c\rangle, \langle d, b\rangle, \langle d, a\rangle, \langle c, b\rangle, \langle c, a\rangle, \langle b, a\rangle\}$

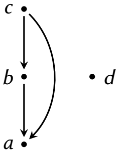
(12)



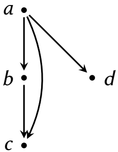
(13) a. $R_2 = \{\langle b, a\rangle, \langle c, b\rangle, \langle c, a\rangle\}$

b. $R_5 = \{\langle a, b\rangle, \langle a, c\rangle, \langle a, d\rangle, \langle b, c\rangle\}$

(14) a.



b.



Cardinality, set equivalence

- the cardinality of a set = the number of members/elements inside that set
- for a given set A, the cardinality of A is written as $|A|$

(15) a. $X = \{a, b, c\}$

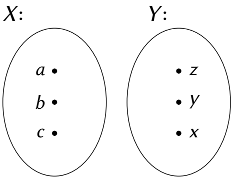
b. $|X| = 3$

- two sets are considered **equivalent** iff there exists a (total) one-to-one correspondence between them
- what would this mean again for sets X and Y below?

(16) a. $X = \{a, b, c\}$

b. $Y = \{x, y, z\}$

(17)



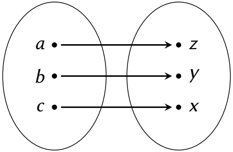
Cardinality, set equivalence

- two sets are considered **equivalent** iff there exists a one-to-one correspondence between them

(18) a. $X = \{a, b, c\}$

b. $Y = \{x, y, z\}$

(19) X: Y:



- since X and Y are equivalent, they are also of the same size

(20) $|X| = |Y| = 3$

- you will also find this notation for set equivalence: $X \sim Y$
- equal vs. equivalence: two sets are equal iff they have the same members; set equivalence has to do with the number of members
- set equivalence is thus a weaker notion than set equality: two equal sets will always also be equivalent; however, two equivalent sets need not be equal

New topic

Logic and Formal Systems

Logic

- logic is the study of reasoning
- in particular, it is concerned with the question which patterns of reasoning are valid and which are not
- we all use logic to reason about the objects in the world we live in
- our reasoning is based on **axioms** — a set of assumptions that we hold to be true in our world
- the axioms we have are relative to the time and place we live in, and may thus change with time
- here are some statements that were axioms in the past, but not anymore:
    1. The earth is the center of the solar system.
    2. Pluto is a planet.
    3. Smoking does not lead to cancer.
    4. Climate change is not real.

Logic

- a formal system, such as a logic, consists of:
  1. a non-empty set of primitives: the things/objects we are interested in investigating further
     e.g. *Pluto, earth, climate change, Pluto is a planet*
  2. a set of statements, called axioms, about those primitives
     e.g. *Pluto is not a planet*; *Madonna is alive and Prince is dead*
  3. a way to reason, i.e. derive further statements from these axioms

(21)
All women are mortals.
Madonna is a woman.
$\therefore$ Madonna is a mortal.

- read $\therefore$ as "therefore", premises (statements) are above the line, the conclusion is below the line

Logic

- we have an intuitive understanding of which reasoning is valid and which is not

(22) All women are mortals.
Madonna is a woman.
$\therefore$ Madonna is a mortal.

(23) Some women are mortals.
Madonna is a woman.
$\not\therefore$ Madonna is a mortal.

- if we accept the truth of the premise of a valid argument, we cannot deny its consequence
- the validity of an argument only depends on its form (and on some of the words that are part of the statement, such as *all* and *some* above)

(24) All X's are Y.
Ψ is an X.
$\therefore$ Ψ is a Y.

(25) Some X's are Y.
Ψ is an X.
$\not\therefore$ Ψ is a Y.

Natural vs. Formal Languages

- languages we speak and use naturally to communicate with each other are what we call natural languages
- formal languages, on the other hand, are usually designed by people for a clear, particular purpose
  - set theory
  - statement logic
  - predicate logic
  - programming languages like Python, Perl, or C++
  - Morse code
- we will be concerned with two types of formal languages: statement logic and predicate logic
- these have been deliberately designed to be syntactically and semantically simpler than natural language and avoid many of its ambiguities and subtleties

Object vs. Meta Language

- it is important that we keep separate two types of language:
  - **object language**:
    language system that is the object of our study
  - **meta language**:
    language system we use to talk about the object language
- we can use set theory, for instance, as a metalanguage to talk about physical systems
- we can also talk in English about English
- in this class we will use statement logic and then later on predicate logic to talk about natural languages such as English, German, and so on . . .

Syntax and semantics of a formal system

- within languages we separate between form and content
  - **syntax**:
    properties of expressions of the system itself, such as its primitives, axioms, rules of inference
  - **semantics**:
    relations between the system and its models or interpretations
- one needs a set of basic expressions, the vocabulary, from which more complex expressions can be built according to the syntactic rules
- finding a model for a theory requires finding . . .
  - some abstract or concrete structured domain and . . .
  - an interpretation for all of the primitive expressions of the theory in that domain . . .

. . . such that on that interpretation, all of the statements (the axioms) in the theory come out true

Statement logic and predicate logic

- statement as well as predicate logic each have their own vocabulary, rules of syntax, and semantics
- the syntactic and semantic components of these languages are very much simpler than those of any natural language
- the sentences of our logical languages are all declaratives – there are no interrogatives (questions), imperatives, performatives, etc.
- the means of joining sentences together to form compound sentences is severely limited:
- in statement logic, we will have sentential connectives corresponding (roughly) to English *and, or, not, if ... then*, and *if and only if*, but nothing to *because, while, after, although, ...*
- in predicate logic we will in addition find counterparts of a few determiners of English: *some, all, no, every*, but not *most, many, a few, several, one half, ...*

Statement logic

- today and for the next two sessions, we will take a closer look at statement logic

Statement logic

A formal system where the primitives are all statements.

- we will assume an infinite vocabulary of atomic statements (atomic in the sense that these sentences are in their simplest form)

(26) *Basic expressions of statement logic*

$$p, q, r, s, p', p'', \ldots$$

- we can think of atomic statements in our formal statement logic system as being like very simple declarative sentences in natural language, e.g. *Climate change is happening.*

Statement logic

- we also have syntactic rules of wellformedness, i.e. we can use atomic statements to create more complex statements or formulas (well-formed formula often abbreviated as *wff*)

(27) *Syntax of statement logic*

a. An atomic statement is a well-formed formula.

b. If ϕ is a well-formed formula, then $(\neg\phi)$ is a well-formed formula.

c. If ϕ and ψ are well-formed formulas, then $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, and $(\phi \leftrightarrow \psi)$ are well-formed formulas.

d. Nothing else is a (well-formed) formula.

- any two distinct *wffs* can be made into one *wff* by means of adding any of (binary) **connectives** between them ($\neg$ = negation, is a unary "connective", of course)
  - $\wedge$ = conjunction, similar to "and"
  - $\vee$ = disjunction, similar to (inclusive) "or"
  - $\rightarrow$ = conditional, similar to "if ... then"
  - $\leftrightarrow$ = biconditional, similar to "if and only if"
- finally, we enclose the result of forming a complex formula in parentheses

Exercise

(28) *Basic expressions of statement logic*

$p, q, r, s, p', p'', \ldots$

(29) *Syntax of statement logic*

a. An atomic statement is a well-formed formula.

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d. Nothing else is a formula.

(30) Determine the well-formedness of the following formulae:

a. p

b. q'

c. $(\neg r)$

d. $\neg$

e. $(r \rightarrow s)$

f. $r \leftrightarrow s$

Exercise

(28) *Basic expressions of statement logic*

$p, q, r, s, p', p'', \ldots$

(29) *Syntax of statement logic*

a. An atomic statement is a well-formed formula.

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d. Nothing else is a formula.

(30) Determine the well-formedness of the following formulae:

a. p **well-formed**

b. q' **well-formed**

c. $(\neg r)$ **well-formed**

d. $\neg$ **not well-formed**

e. $(r \rightarrow s)$ **well-formed**

f. $r \leftrightarrow s$ **not well-formed**

Statement logic

- we said that atomic statements correspond to simple declarative sentences
- but such sentences can also look more complex while still being considered an atomic statement

(31) a. Climate change is happening.

b. The government's constant denial of climate change has led Mary to consider participating in a demonstration this Friday.

- statement logic also ignores pragmatic concerns like speaker, hearer, utterance context, or information packaging
- two distinct sentences in natural language which may differ from one another in pragmatics but have the same meaning would count as the same statement in statement logic

(32) a. Wasting food is one of the problems of modern society.

b. One of the problems of modern society is wasting food.

Statement logic

- now let us talk about the semantics of statement logic
- statements, both atomic and complex, all have a truth value
- since statement logic is a binary logic, there are two truth values: True (or 1) and False (or 0), no other option is possible
- the truth value of an atomic sentence, say p, depends on nothing other than the content of p (this will mostly be determined by our model)
- the truth value of a complex statement (built out of atomic statements) like: $((\neg p) \wedge q)$ will depend on:
  - the truth value of p
  - the truth value of q
  - the truth functional properties of the connectives $\neg$ and $\wedge$
- unlike the truth values of statements, which vary according to the content of these statements (the model), the truth functional properties of connectives are uniform and universal

Statement logic: negation

- the easiest way to illustrate the semantics of statement logic are truth tables; we will start with negation
- if p is True, than negating p produces the truth value False (and vice versa)

(33) Truth table for negation:

p	$(\neg p)$
1	0
0	1

- English often expresses negation with “not”, sometimes with an additional auxiliary

(34) a. John is here vs. John is **not** here p vs. $(\neg p)$

b. John smokes vs. John **does not** smoke p vs. $(\neg p)$

Statement logic: conjunction

- conjunction $\wedge$ is very close to English "and"
- imagine the following contents for p and q and try to fill in the truth values in the table!

(35) a. p = John smokes

b. q = Jane snores

(36) Truth table for conjunction:

p	q	$(p \wedge q)$
1	1	
1	0	
0	1	
0	0	

Statement logic: conjunction

- conjunction $\wedge$ is very close to English "and"
- imagine the following contents for p and q and try to fill in the truth values in the table!

(35) a. p = John smokes

b. q = Jane snores

(36) Truth table for conjunction:

p	q	$(p \wedge q)$
1	1	1
1	0	0
0	1	0
0	0	0

- essentially, a complex expression: $(p \wedge q)$ has a truth value of True iff both p and q are True, in all other cases it is False

Statement logic: disjunction

- disjunction $\vee$ corresponds to one particular use of English "or"
- imagine the following contents for p and q and try to fill in the truth values in the table!

(37) To watch this movie, ...

a. p = you have to be over 13

b. q = be accompanied by a parent

(38) Truth table for disjunction:

p	q	$(p \vee q)$
1	1	
1	0	
0	1	
0	0	

Statement logic: disjunction

- disjunction $\vee$ corresponds to one particular use of English "or"
- imagine the following contents for p and q and try to fill in the truth values in the table!

(37) To watch this movie, . . .

a. p = you have to be over 13

b. q = be accompanied by a parent

(38) Truth table for disjunction:

p	q	$(p \vee q)$
1	1	1
1	0	1
0	1	1
0	0	0

- for $(p \vee q)$ to be True at least one (including both), p or q, has to be True, otherwise it is False; this is the **inclusive** disjunction

Statement logic: disjunction

- there is also another prominent interpretation of English "or"
- which truth value does not fit anymore in the truth table given the following context?

(39) I cannot remember what Peter had to drink last time, ...

a. p = Peter had tea

b. q = Peter had coffee

(40) Truth table for disjunction:

p	q	$(p \vee q)$
1	1	1
1	0	1
0	1	1
0	0	0

- the first row would be 0!
- this is the **exclusive** reading of "or" which can also be made explicit with "either ... or" (statement logic only recognizes inclusive disjunction)

Statement logic: conditional

- the natural language correpondent for the conditional $\rightarrow$ is "if ... then"
- the first two rows of the table are easy to understand with the example below:

(41) a. p = Mary is at the party

b. q = John is at the party

(42) Truth table for conditional:

p	q	$(p \rightarrow q)$
1	1	1
1	0	0
0	1	1
0	0	1

- clearly when p is True but q is False, the conditional does not hold, hence it is False
- the last two rows show what happens when p is False: one may be reluctant to say that the conditional is False, rather, it does not seem to have a truth value
- but in a two-valued logic (such as statement logic), if a statement is not False, it must be true!
- this reasoning makes the last two rows (of the conditional) True

Statement logic: biconditional

- the biconditional $\leftrightarrow$ corresponds to English "if and only if" (iff)
- here is an example

(43) a. p = I will leave tomorrow

b. q = I get the car fixed

(44) Truth table for biconditional:

p	q	$(p \leftrightarrow q)$
1	1	1
1	0	0
0	1	0
0	0	1

- $(p \leftrightarrow q)$ is the same as ("logically equivalent to", see next session) $((p \rightarrow q) \wedge (q \rightarrow p))$, which also explains the third row