# Formale Grundlagen (Logik)
# Modul 04-006-1001

## Statement Logic IV

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(Slides by Imke Driemel & Sandhya Sundaresan,
based on Partee, ter Meulen und Wall 1990
"Mathematical Methods in Linguistics")

# Recap: Formal components of a proof

- constructing a proof/argument consists of two parts
  1. a number of statements, called **premises**: these are just statements that we, for the sake of argument, assume to be True
  2. a **conclusion**, whose truth is demonstrated to necessarily follow from the assumed truth of the premises

(1)

$$\begin{array}{ll} & \text{premise 1} \\ & \text{premise 2} \\ & \dots \\ \hline \therefore & \text{conclusion} \end{array}$$

- a proof is called **valid** iff there is no uniform assignment of truth values to its atomic statements which makes all its premises true and its conclusion false
- a proof is called **invalid** iff there is at least one uniform assignment of truth values to its atomic statements which makes all its premises true and its conclusion false
- premises and conclusion of a proof are related by the conditional $\rightarrow$ (premise 1 $\wedge$ premise 2 $\wedge \ldots \rightarrow$ conclusion)

# Recap: Proofs

- let us remind ourselves of this based on the proof **Modus Tollens**

(2)
If Jack drinks beer, he will get drunk.
Jack doesn't get drunk.
∴ Jack doesn't drink beer.

(3)
$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

- we show the validity of the proof with a truth table (all values are True in the last column)

(4)

| $p$ | $q$ | $(p \rightarrow q)$ | $((p \rightarrow q) \wedge (\neg q))$ | $(((p \rightarrow q) \wedge (\neg q)) \rightarrow (\neg p))$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 | 1 |
| 0 | 0 | 1 | 1 | 1 |

# Recap: Fallacies (invalid proofs)

- example of an invalid proof: **fallacy of affirming the consequent**

(5)
If Marie eats another pizza, she will get sick.
Marie gets sick.
$\therefore\!\!\!/$ Marie eats another pizza.

(6)
$(p \rightarrow q)$
$q$
$\therefore\!\!\!/$ $p$

- we can show that the proof is invalid with a truth table (not all values are True in the last column)

(7)

| $p$ | $q$ | $(p \rightarrow q)$ | $((p \rightarrow q) \wedge q)$ | $(((p \rightarrow q) \wedge q) \rightarrow p)$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 |
| 0 | 1 | 1 | 1 | 0 |
| 0 | 0 | 1 | 0 | 1 |

# Simple proofs

(8) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(9) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(10) **Hypothetical Syllogism**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(11) **Disjunctive Syllogism**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(12) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(13) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(14) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

# Simple proofs

(8) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(9) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(10) **Hypothetical Syllogism**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(11) **Disjunctive Syllogism**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(12) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(13) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(14) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

Given the premises 1.-5. we can prove $t$ !

(15) *simple proof*:

1. $(p \rightarrow q)$
2. $(p \vee s)$
3. $(q \rightarrow r)$
4. $(s \rightarrow t)$
5. $(\neg r)$
6. 
7. 
8. 
9. $t$

# Simple proofs

(8) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(9) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(10) **Hypothetical Syllogism**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(11) **Disjunctive Syllogism**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(12) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(13) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(14) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

Given the premises 1.-5. we can prove $t$ !

(15) *simple proof*:

| | | |
|---|---|---|
| 1. | $(p \rightarrow q)$ | |
| 2. | $(p \vee s)$ | |
| 3. | $(q \rightarrow r)$ | |
| 4. | $(s \rightarrow t)$ | |
| 5. | $(\neg r)$ | |
| 6. | $(\neg q)$ | 3,5 MT |
| 7. | $(\neg p)$ | 1,6 MT |
| 8. | $s$ | 2,7 DS |
| 9. | $t$ | 4,8 MP |

# Simple proofs: Reverse engineering

(16) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(17) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(18) **Hypothetical Syllogism**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(19) **Disjunctive Syllogism**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(20) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(21) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(22) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*$t$ is part of 4.*
*With which proof can we move from 4. to $t$?*

Given the premises 1.-5. we can prove $t$ !

(23) *simple proof*:

1. $(p \rightarrow q)$
2. $(p \vee s)$
3. $(q \rightarrow r)$
4. $(s \rightarrow t)$
5. $(\neg r)$
6.
7.
8.
9. $t$

# Simple proofs: Reverse engineering

(16) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(17) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(18) **Hypothetical Syllogism**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(19) **Disjunctive Syllogism**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(20) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(21) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(22) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*$t$ is part of 4.*
*With which proof can we move from 4. to $t$?*
*With MP, but only if $s$ is true!*

Given the premises 1.-5. we can prove $t$ !

(23) *simple proof*:

1. $(p \rightarrow q)$
2. $(p \vee s)$
3. $(q \rightarrow r)$
4. $(s \rightarrow t)$
5. $(\neg r)$
6.
7.
8. $s$
9. $t$ 4,8 MP

# Simple proofs: Reverse engineering

(24) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(25) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(26) **Hypothetical Syllogism**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(27) **Disjunctive Syllogism**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(28) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(29) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(30) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*$s$ is part of 2.*
*With which proof can we move from 2. to $s$?*

Given the premises 1.-5. we can prove $t$ !

(31) *simple proof*:

1. $(p \rightarrow q)$
2. $(p \vee s)$
3. $(q \rightarrow r)$
4. $(s \rightarrow t)$
5. $(\neg r)$
6.
7.
8. $s$
9. $t$ 4,8 MP

# Simple proofs: Reverse engineering

(24) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(25) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(26) **Hypothetical Syllogism**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(27) **Disjunctive Syllogism**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(28) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(29) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(30) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*$s$ is part of 2.*
*With which proof can we move from 2. to $s$?*
*With DS, but only if $(\neg p)$ is true!*

Given the premises 1.-5. we can prove $t$ !

(31) *simple proof*:

1. $(p \rightarrow q)$
2. $(p \vee s)$
3. $(q \rightarrow r)$
4. $(s \rightarrow t)$
5. $(\neg r)$
6. 
7. $(\neg p)$
8. $s$ — 2,7 DS
9. $t$ — 4,8 MP

# Simple proofs: Reverse engineering

(32) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(33) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(34) **Hypothetical Syllogism**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(35) **Disjunctive Syllogism**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(36) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(37) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(38) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*$p$ is part of 1. and 2.*
*With which proof can we move from either 2. or 1. to $(\neg p)$?*

Given the premises 1.-5. we can prove $t$ !

(39) *simple proof*:

1. $(p \rightarrow q)$
2. $(p \vee s)$
3. $(q \rightarrow r)$
4. $(s \rightarrow t)$
5. $(\neg r)$
6. 
7. $(\neg p)$
8. $s$ — 2,7 DS
9. $t$ — 4,8 MP

# Simple proofs: Reverse engineering

(32) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(33) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(34) **Hypothetical Syllogism**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(35) **Disjunctive Syllogism**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(36) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(37) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(38) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*$p$ is part of 1. and 2.*
*With which proof can we move from either 2. or 1. to $(\neg p)$?*
*With MT, but only if $(\neg q)$ is true!*

Given the premises 1.-5. we can prove $t$ !

(39) *simple proof*:

| | | |
|---|---|---|
| 1. | $(p \rightarrow q)$ | |
| 2. | $(p \vee s)$ | |
| 3. | $(q \rightarrow r)$ | |
| 4. | $(s \rightarrow t)$ | |
| 5. | $(\neg r)$ | |
| 6. | $(\neg q)$ | |
| 7. | $(\neg p)$ | 1,6 MT |
| 8. | $s$ | 2,7 DS |
| 9. | $t$ | 4,8 MP |

# Simple proofs: Reverse engineering

(40) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(41) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(42) **Hypothetical Syllogism**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(43) **Disjunctive Syllogism**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(44) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(45) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(46) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*Does* $(\neg q)$ *follow from any of the premises 1.-5.?*

Given the premises 1.-5. we can prove $t$ !

(47) *simple proof*:

| | | |
|---|---|---|
| 1. | $(p \rightarrow q)$ | |
| 2. | $(p \vee s)$ | |
| 3. | $(q \rightarrow r)$ | |
| 4. | $(s \rightarrow t)$ | |
| 5. | $(\neg r)$ | |
| 6. | $(\neg q)$ | |
| 7. | $(\neg p)$ | 1,6 MT |
| 8. | $s$ | 2,7 DS |
| 9. | $t$ | 4,8 MP |

# Simple proofs: Reverse engineering

(40) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(41) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(42) **Hypothetical Syllogism**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(43) **Disjunctive Syllogism**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(44) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(45) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(46) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*Does* $(\neg q)$ *follow from any of the premises 1.-5.?*
*Yes, with MT, we can conclude* $(\neg q)$ *from 3. and 5.!*

Given the premises 1.-5. we can prove $t$ !

(47) *simple proof*:

| | | |
|---|---|---|
| 1. | $(p \rightarrow q)$ | |
| 2. | $(p \vee s)$ | |
| 3. | $(q \rightarrow r)$ | |
| 4. | $(s \rightarrow t)$ | |
| 5. | $(\neg r)$ | |
| 6. | $(\neg q)$ | 3,5 MT |
| 7. | $(\neg p)$ | 1,6 MT |
| 8. | $s$ | 2,7 DS |
| 9. | $t$ | 4,8 MP |

# Complex proofs

(48) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(49) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(50) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(51) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(52) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(53) **Identity Laws**:

a. $x \vee \mathit{False} \Leftrightarrow x$

b. $x \wedge \mathit{False} \Leftrightarrow \mathit{False}$

c. $x \vee \mathit{True} \Leftrightarrow \mathit{True}$

d. $x \wedge \mathit{True} \Leftrightarrow x$

(54) **Conditional Laws**:

a. $(p \rightarrow q) \Leftrightarrow ((\neg p) \vee q)$

b. $(p \rightarrow q) \Leftrightarrow ((\neg q) \rightarrow (\neg p))$

(55) **Commutative Laws**:

a. $x \vee y \Leftrightarrow y \vee x$

b. $x \wedge y \Leftrightarrow y \wedge x$

(56) **Associative Laws**:

a. $(x \vee y) \vee z \Leftrightarrow x \vee (y \vee z)$

b. $(x \wedge y) \wedge z \Leftrightarrow x \wedge (y \wedge z)$

Given the premises 1.-2. we can prove $(p \rightarrow q)$ !

(57) *complex proof*:

| | | |
|---|---|---|
| 1. | $(p \rightarrow (q \vee r))$ | |
| 2. | $(\neg r)$ | |
| 3. | $((\neg p) \vee (q \vee r))$ | 1 Cond |
| 4. | $(((\neg p) \vee q) \vee r)$ | 3 Ass |
| 5. | $(((\neg p) \vee q) \vee F)$ | 4 Neg |
| 6. | $(((\neg p) \vee q))$ | 5 Ident |
| 7. | $(p \rightarrow q)$ | 6 Cond |

# Complex proofs: Reverse engineering

(58) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(59) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(60) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(61) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(62) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(63) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(64) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*Is there a proof such that we can conclude $(p \rightarrow q)$ given our premises?*

Given the premises 1.-2. we can prove $(p \rightarrow q)$ !

(65) *complex proof*:

1. $(p \rightarrow (q \vee r))$
2. $(\neg r)$
3. 
4. 
5. 
6. 
7. $(p \rightarrow q)$

# Complex proofs: Reverse engineering

(58) **Modus Ponens**

$$\begin{array}{ll} & (p \to q) \\ & p \\ \hline \therefore & q \end{array}$$

(59) **Modus Tollens**

$$\begin{array}{ll} & (p \to q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(60) **Hyp. Syll.**

$$\begin{array}{ll} & (p \to q) \\ & (q \to r) \\ \hline \therefore & (p \to r) \end{array}$$

(61) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(62) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(63) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(64) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*Is there a proof such that we can conclude $(p \to q)$ given our premises?*
*Not really . . . we have to make use of equivalence laws to substitute our conclusion into a formula we can work with.*

Given the premises 1.-2. we can prove $(p \to q)$ !

(65) *complex proof*:

1. $(p \to (q \vee r))$
2. $(\neg r)$
3. 
4. 
5. 
6. 
7. $(p \to q)$

# Complex proofs: Reverse engineering

(66) **Idempotent Laws**:

a. $(x \vee x) \Leftrightarrow x$

b. $(x \wedge x) \Leftrightarrow x$

(67) **Complement Laws**:

a. $p \vee (\neg p) \Leftrightarrow True$

b. $\neg(\neg p) \Leftrightarrow p$

c. $p \wedge (\neg p) \Leftrightarrow False$

(68) **Identity Laws**:

a. $x \vee False \Leftrightarrow x$

b. $x \wedge False \Leftrightarrow False$

c. $x \vee True \Leftrightarrow True$

d. $x \wedge True \Leftrightarrow x$

(69) **Commutative Laws**:

a. $x \vee y \Leftrightarrow y \vee x$

b. $x \wedge y \Leftrightarrow y \wedge x$

(70) **Distributive Laws**:

a. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

b. $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

(71) **DeMorgan's Laws**:

a. $\neg(p \vee q) \Leftrightarrow (\neg p) \wedge (\neg q)$

b. $\neg(p \wedge q) \Leftrightarrow (\neg p) \vee (\neg q)$

(72) **Conditional Laws**:

a. $(p \rightarrow q) \Leftrightarrow ((\neg p) \vee q)$

b. $(p \rightarrow q) \Leftrightarrow ((\neg q) \rightarrow (\neg p))$

(73) **Associative Laws**:

a. $(x \vee y) \vee z \Leftrightarrow x \vee (y \vee z)$

b. $(x \wedge y) \wedge z \Leftrightarrow x \wedge (y \wedge z)$

*Which laws can we apply to $(p \rightarrow q)$?*

Given the premises 1.-2. we can prove $(p \rightarrow q)$ !

(74) *complex proof*:

1. $(p \rightarrow (q \vee r))$
2. $(\neg r)$
3. 
4. 
5. 
6. 
7. $(p \rightarrow q)$

# Complex proofs: Reverse engineering

(66) **Idempotent Laws**:

a. $(x \vee x) \Leftrightarrow x$

b. $(x \wedge x) \Leftrightarrow x$

(67) **Complement Laws**:

a. $p \vee (\neg p) \Leftrightarrow True$

b. $\neg(\neg p) \Leftrightarrow p$

c. $p \wedge (\neg p) \Leftrightarrow False$

(68) **Identity Laws**:

a. $x \vee False \Leftrightarrow x$

b. $x \wedge False \Leftrightarrow False$

c. $x \vee True \Leftrightarrow True$

d. $x \wedge True \Leftrightarrow x$

(69) **Commutative Laws**:

a. $x \vee y \Leftrightarrow y \vee x$

b. $x \wedge y \Leftrightarrow y \wedge x$

(70) **Distributive Laws**:

a. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

b. $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

(71) **DeMorgan's Laws**:

a. $\neg(p \vee q) \Leftrightarrow (\neg p) \wedge (\neg q)$

b. $\neg(p \wedge q) \Leftrightarrow (\neg p) \vee (\neg q)$

(72) **Conditional Laws**:

a. $(p \rightarrow q) \Leftrightarrow ((\neg p) \vee q)$

b. $(p \rightarrow q) \Leftrightarrow ((\neg q) \rightarrow (\neg p))$

(73) **Associative Laws**:

a. $(x \vee y) \vee z \Leftrightarrow x \vee (y \vee z)$

b. $(x \wedge y) \wedge z \Leftrightarrow x \wedge (y \wedge z)$

*Which laws can we apply to $(p \rightarrow q)$?*
*Conditional laws! The first one seems more promising . . .*

Given the premises 1.-2. we can prove $(p \rightarrow q)$ !

(74) *complex proof*:

1. $(p \rightarrow (q \vee r))$
2. $(\neg r)$
3. 
4. 
5. 
6. $((\neg p) \vee q)$
7. $(p \rightarrow q)$ 6 Cond

# Complex proofs: Reverse engineering

(75) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(76) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(77) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(78) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(79) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(80) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(81) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*The next proof is difficult to see. We will derive 6. by applying DS with the help of premise 2. What would be the other statement we have to create?*

Given the premises 1.-2. we can prove $(p \rightarrow q)$ !

(82) *complex proof*:

1. $(p \rightarrow (q \vee r))$
2. $(\neg r)$
3.
4.
5.
6. $((\neg p) \vee q)$
7. $(p \rightarrow q)$ 6 Cond

# Complex proofs: Reverse engineering

(75) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(76) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(77) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(78) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(79) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(80) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(81) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*The next proof is difficult to see. We will derive 6. by applying DS with the help of premise 2. What would be the other statement we have to create?*
*We have to create the disjunction which is the first premise of DS!*

Given the premises 1.-2. we can prove $(p \rightarrow q)$ !

(82) *complex proof*:

1. $(p \rightarrow (q \vee r))$
2. $(\neg r)$
3. 
4. 
5. $(r \vee ((\neg p) \vee q))$
6. $((\neg p) \vee q)$ 2,5 DS
7. $(p \rightarrow q)$ 6 Cond

# Complex proofs: Reverse engineering

(83) **Idempotent Laws**:

a. $(x \vee x) \Leftrightarrow x$

b. $(x \wedge x) \Leftrightarrow x$

(84) **Complement Laws**:

a. $p \vee (\neg p) \Leftrightarrow True$

b. $\neg(\neg p) \Leftrightarrow p$

c. $p \wedge (\neg p) \Leftrightarrow False$

(85) **Identity Laws**:

a. $x \vee False \Leftrightarrow x$

b. $x \wedge False \Leftrightarrow False$

c. $x \vee True \Leftrightarrow True$

d. $x \wedge True \Leftrightarrow x$

(86) **Commutative Laws**:

a. $x \vee y \Leftrightarrow y \vee x$

b. $x \wedge y \Leftrightarrow y \wedge x$

(87) **Distributive Laws**:

a. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

b. $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

(88) **DeMorgan's Laws**:

a. $\neg(p \vee q) \Leftrightarrow (\neg p) \wedge (\neg q)$

b. $\neg(p \wedge q) \Leftrightarrow (\neg p) \vee (\neg q)$

(89) **Conditional Laws**:

a. $(p \rightarrow q) \Leftrightarrow ((\neg p) \vee q)$

b. $(p \rightarrow q) \Leftrightarrow ((\neg q) \rightarrow (\neg p))$

(90) **Associative Laws**:

a. $(x \vee y) \vee z \Leftrightarrow x \vee (y \vee z)$

b. $(x \wedge y) \wedge z \Leftrightarrow x \wedge (y \wedge z)$

*How does that help us?*

Given the premises 1.-2. we can prove $(p \rightarrow q)$ !

(91) *complex proof*:

1. $(p \rightarrow (q \vee r))$
2. $(\neg r)$
3. 
4. 
5. $(r \vee ((\neg p) \vee q))$
6. $((\neg p) \vee q)$ 2,5 DS
7. $(p \rightarrow q)$ 6 Cond

# Complex proofs: Reverse engineering

(83) **Idempotent Laws**:

a. $(x \vee x) \Leftrightarrow x$

b. $(x \wedge x) \Leftrightarrow x$

(84) **Complement Laws**:

a. $p \vee (\neg p) \Leftrightarrow \mathit{True}$

b. $\neg(\neg p) \Leftrightarrow p$

c. $p \wedge (\neg p) \Leftrightarrow \mathit{False}$

(85) **Identity Laws**:

a. $x \vee \mathit{False} \Leftrightarrow x$

b. $x \wedge \mathit{False} \Leftrightarrow \mathit{False}$

c. $x \vee \mathit{True} \Leftrightarrow \mathit{True}$

d. $x \wedge \mathit{True} \Leftrightarrow x$

(86) **Commutative Laws**:

a. $x \vee y \Leftrightarrow y \vee x$

b. $x \wedge y \Leftrightarrow y \wedge x$

(87) **Distributive Laws**:

a. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

b. $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

(88) **DeMorgan's Laws**:

a. $\neg(p \vee q) \Leftrightarrow (\neg p) \wedge (\neg q)$

b. $\neg(p \wedge q) \Leftrightarrow (\neg p) \vee (\neg q)$

(89) **Conditional Laws**:

a. $(p \rightarrow q) \Leftrightarrow ((\neg p) \vee q)$

b. $(p \rightarrow q) \Leftrightarrow ((\neg q) \rightarrow (\neg p))$

(90) **Associative Laws**:

a. $(x \vee y) \vee z \Leftrightarrow x \vee (y \vee z)$

b. $(x \wedge y) \wedge z \Leftrightarrow x \wedge (y \wedge z)$

*How does that help us?*
*It turns out that we can simplify 1. to 5.!*

Given the premises 1.-2. we can prove $(p \rightarrow q)$ !

(91) *complex proof*:

1. $(p \rightarrow (q \vee r))$
2. $(\neg r)$
3. $((\neg p) \vee (q \vee r))$ 1 Cond
4. $(((\neg p) \vee q) \vee r)$ 3 Ass
5. $(r \vee ((\neg p) \vee q))$ 4 Comm
6. $((\neg p) \vee q)$ 2,5 DS
7. $(p \rightarrow q)$ 6 Cond

# Direct conditional proofs

- complex proofs such as the one we just went through are tricky
- fortunately, there is a simpler method for such a proof, i.e. a proof where the conclusion is a conditional
- if the conclusion of a proof contains a conditional as the main connective, we can use a method of argumentation called **conditional proof**
  - suppose a proof has premises: $p_1, p_2, \dots p_n$ and $(q \rightarrow r)$ as the conclusion
  - in the conditional proof, we add the antecedent $q$ of the conclusion as an additional **auxiliary premise**
  - we then derive $r$ from the premises $p_1, p_2, \dots p_n$ and the auxiliary premise $q$
- the validity of the conditional proof is based on the following logical equivalence:

(92) $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow (q \rightarrow r) \Leftrightarrow (p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge q) \rightarrow r$

# Direct conditional proofs

- a conditional proof begins with the assumption that the antecedent is true, then logical reasoning is used to show that the consequent must also be true (given other premises)
- a conditional proof shows that the antecedent implies the consequent

Given the premises 1.-2. we can prove $(p \rightarrow q)$ !

(93) *complex proof*:

| | | |
|---|---|---|
| 1. | $(p \rightarrow (q \vee r))$ | |
| 2. | $(\neg r)$ | |
| 3. | $((\neg p) \vee (q \vee r))$ | 1 Cond |
| 4. | $(((\neg p) \vee q) \vee r)$ | 3 Ass |
| 5. | $(r \vee ((\neg p) \vee q))$ | 4 Comm |
| 6. | $((\neg p) \vee q)$ | 2,5 DS |
| 7. | $(p \rightarrow q)$ | 6 Cond |

Given the premises 1.-2. we can prove $(p \rightarrow q)$ !

(94) *conditional proof*:

| | | |
|---|---|---|
| 1. | $(p \rightarrow (q \vee r))$ | |
| 2. | $(\neg r)$ | |
| 3. | │ $p$ | Aux |
| 4. | │ | |
| 5. | │ | |
| 6. | │ $q$ | |
| 7. | $(p \rightarrow q)$ | 3-6 CP |

# Direct conditional proofs

(95) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(96) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(97) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(98) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(99) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(100) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(101) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*What are the next steps?*

Given the premises 1.-2. we can prove $(p \rightarrow q)$ !

(102) *conditional proof*:

| | | |
|---|---|---|
| 1. | $(p \rightarrow (q \vee r))$ | |
| 2. | $(\neg r)$ | |
| 3. | \| $p$ | Aux |
| 4. | \| | |
| 5. | \| | |
| 6. | \| $q$ | |
| 7. | $(p \rightarrow q)$ | 3-6 CP |

# Direct conditional proofs

(95) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(96) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(97) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(98) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(99) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(100) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(101) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*What are the next steps?*

Given the premises 1.-2. we can prove $(p \rightarrow q)$ !

(102) *conditional proof*:

| | | |
|---|---|---|
| 1. | $(p \rightarrow (q \vee r))$ | |
| 2. | $(\neg r)$ | |
| 3. | │ $p$ | Aux |
| 4. | │ $(q \vee r)$ | 1,3 MP |
| 5. | │ | |
| 6. | │ $q$ | |
| 7. | $(p \rightarrow q)$ | 3-6 CP |

# Direct conditional proofs

(95) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(96) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(97) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(98) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(99) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(100) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(101) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*What are the next steps?*

Given the premises 1.-2. we can prove $(p \rightarrow q)$ !

(102) *conditional proof*:

| | | |
|---|---|---|
| 1. | $(p \rightarrow (q \vee r))$ | |
| 2. | $(\neg r)$ | |
| 3. | \| $p$ | Aux |
| 4. | \| $(q \vee r)$ | 1,3 MP |
| 5. | \| $(r \vee q)$ | 4 Comm |
| 6. | \| $q$ | |
| 7. | $(p \rightarrow q)$ | 3-6 CP |

# Direct conditional proofs

(95) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(96) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(97) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(98) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(99) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(100) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(101) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*What are the next steps?*

Given the premises 1.-2. we can prove $(p \rightarrow q)$ !

(102) *conditional proof*:

| | | |
|---|---|---|
| 1. | $(p \rightarrow (q \vee r))$ | |
| 2. | $(\neg r)$ | |
| 3. | $\quad \vert\ p$ | Aux |
| 4. | $\quad \vert\ (q \vee r)$ | 1,3 MP |
| 5. | $\quad \vert\ (r \vee q)$ | 4 Comm |
| 6. | $\quad \vert\ q$ | 2,5 DS |
| 7. | $(p \rightarrow q)$ | 3-6 CP |

# Direct conditional proofs

- conditional proofs are called conditional because they rely on an additional premise
- the conclusion is true under the condition that the auxiliary premise is true

Given the premises 1.-2. we can prove $(p \rightarrow q)$ !

(103) *complex proof*:

| | | |
|---|---|---|
| 1. | $(p \rightarrow (q \vee r))$ | |
| 2. | $(\neg r)$ | |
| 3. | $((\neg p) \vee (q \vee r))$ | 1 Cond |
| 4. | $(((\neg p) \vee q) \vee r)$ | 3 Ass |
| 5. | $(r \vee ((\neg p) \vee q))$ | 4 Comm |
| 6. | $((\neg p) \vee q)$ | 2,5 DS |
| 7. | $(p \rightarrow q)$ | 6 Cond |

Given the premises 1.-2. we can prove $(p \rightarrow q)$ !

(104) *conditional proof*:

| | | |
|---|---|---|
| 1. | $(p \rightarrow (q \vee r))$ | |
| 2. | $(\neg r)$ | |
| 3. | \| $p$ | Aux |
| 4. | \| $(q \vee r)$ | 1,3 MP |
| 5. | \| $(r \vee q)$ | 4 Comm |
| 6. | \| $q$ | 2,5 DS |
| 7. | $(p \rightarrow q)$ | 3-6 CP |

# Direct conditional proofs

- conditional proofs are called conditional because they rely on additional premise
- the conclusion is true under the condition that the auxiliary assumption is true

  - we indicate with a vertical bar every line of the proof which is based on the auxiliary premise
  - this is to remind ourselves that we are working with an additional special assumption
  - it is very important to always cancel that auxiliary premise by the rule of Conditional Proof before ending the entire proof (means going back to original position)
  - under the assumption that $p$ is true, the conditional $(p \to q)$ holds
  - $(p \to q)$ does not follow directly from premises 1. and 2.

Given the premises 1.-2. we can prove $(p \to q)$ !

(105) *conditional proof*:

| | | |
|---|---|---|
| 1. | $(p \to (q \lor r))$ | |
| 2. | $(\neg r)$ | |
| 3. | \| $p$ | Aux |
| 4. | \| $(q \lor r)$ | 1,3 MP |
| 5. | \| $(r \lor q)$ | 4 Comm |
| 6. | \| $q$ | 2,5 DS |
| 7. | $(p \to q)$ | 3-6 CP |

# Nested conditional proofs

- a conditional proof can be more complicated with two levels of embedding
  - two auxiliary premises = two vertical bars, one inside the other where the inner bar depends on the outer bar
  - we can always use a statement from a higher level in a lower level
  - but we may not use a statement from a lower level in a higher level
  - under the assumption that $p$ is true, the conditional $(p \rightarrow s)$ holds
  - under the assumption that $(q \rightarrow s)$ is true, the conditional $((q \rightarrow s) \rightarrow (p \rightarrow s))$ holds

Given the premise 1. we can prove $((q \rightarrow s) \rightarrow (p \rightarrow s))$ !

(106) *conditional proof*:

| | | |
|---|---|---|
| 1. | $(p \rightarrow (q \wedge r))$ | |
| 2. | $\mid (q \rightarrow s)$ | Aux |
| 3. | $\mid \mid p$ | Aux |
| 4. | $\mid \mid$ | |
| 5. | $\mid \mid$ | |
| 6. | $\mid \mid s$ | |
| 7. | $\mid (p \rightarrow s)$ | 3-6 CP |
| 8. | $((q \rightarrow s) \rightarrow (p \rightarrow s))$ | 2-7 CP |

# Nested conditional proofs

(107) **Modus Ponens**

$$\begin{array}{ll} & (p \to q) \\ & p \\ \hline \therefore & q \end{array}$$

(108) **Modus Tollens**

$$\begin{array}{ll} & (p \to q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(109) **Hyp. Syll.**

$$\begin{array}{ll} & (p \to q) \\ & (q \to r) \\ \hline \therefore & (p \to r) \end{array}$$

(110) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(111) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(112) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(113) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*What are the next steps?*

Given the premise 1. we can prove $((q \to s) \to (p \to s))$ !

(114) *conditional proof*:

| | | |
|---|---|---|
| 1. | $(p \to (q \wedge r))$ | |
| 2. | $\quad \mid (q \to s)$ | Aux |
| 3. | $\quad \mid \quad \mid p$ | Aux |
| 4. | $\quad \mid \quad \mid$ | |
| 5. | $\quad \mid \quad \mid$ | |
| 6. | $\quad \mid \quad \mid s$ | |
| 7. | $\quad \mid (p \to s)$ | 3-6 CP |
| 8. | $((q \to s) \to (p \to s))$ | 2-7 CP |

# Nested conditional proofs

(107) **Modus Ponens**

$$\begin{array}{ll} & (p \to q) \\ & p \\ \hline \therefore & q \end{array}$$

(108) **Modus Tollens**

$$\begin{array}{ll} & (p \to q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(109) **Hyp. Syll.**

$$\begin{array}{ll} & (p \to q) \\ & (q \to r) \\ \hline \therefore & (p \to r) \end{array}$$

(110) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(111) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(112) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(113) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*What are the next steps?*

Given the premise 1. we can prove $((q \to s) \to (p \to s))$ !

(114) *conditional proof*:

| | | |
|---|---|---|
| 1. | $(p \to (q \wedge r))$ | |
| 2. | $\mid (q \to s)$ | Aux |
| 3. | $\mid \mid p$ | Aux |
| 4. | $\mid \mid (q \wedge r)$ | 1,3 MP |
| 5. | $\mid \mid$ | |
| 6. | $\mid \mid s$ | |
| 7. | $\mid (p \to s)$ | 3-6 CP |
| 8. | $((q \to s) \to (p \to s))$ | 2-7 CP |

# Nested conditional proofs

(107) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(108) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(109) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(110) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(111) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(112) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(113) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*What are the next steps?*

Given the premise 1. we can prove $((q \rightarrow s) \rightarrow (p \rightarrow s))$ !

(114) *conditional proof*:

| | | |
|---|---|---|
| 1. | $(p \rightarrow (q \wedge r))$ | |
| 2. | \| $(q \rightarrow s)$ | Aux |
| 3. | \| \| $p$ | Aux |
| 4. | \| \| $(q \wedge r)$ | 1,3 MP |
| 5. | \| \| $q$ | 4 Simpl |
| 6. | \| \| $s$ | |
| 7. | \| $(p \rightarrow s)$ | 3-6 CP |
| 8. | $((q \rightarrow s) \rightarrow (p \rightarrow s))$ | 2-7 CP |

# Nested conditional proofs

(107) **Modus Ponens**

$$\frac{(p \rightarrow q) \quad p}{\therefore \quad q}$$

(108) **Modus Tollens**

$$\frac{(p \rightarrow q) \quad (\neg q)}{\therefore \quad (\neg p)}$$

(109) **Hyp. Syll.**

$$\frac{(p \rightarrow q) \quad (q \rightarrow r)}{\therefore \quad (p \rightarrow r)}$$

(110) **Dis. Syll.**

$$\frac{(p \vee q) \quad (\neg p)}{\therefore \quad q}$$

(111) **Simplification**

$$\frac{(p \wedge q)}{\therefore \quad p}$$

(112) **Conjunction**

$$\frac{p \quad q}{\therefore \quad (p \wedge q)}$$

(113) **Addition**

$$\frac{p}{\therefore \quad (p \vee q)}$$

*What are the next steps?*

Given the premise 1. we can prove $((q \rightarrow s) \rightarrow (p \rightarrow s))$ !

(114) *conditional proof*:

| | | |
|---|---|---|
| 1. | $(p \rightarrow (q \wedge r))$ | |
| 2. | | $(q \rightarrow s)$ | Aux |
| 3. | | | $p$ | Aux |
| 4. | | | $(q \wedge r)$ | 1,3 MP |
| 5. | | | $q$ | 4 Simpl |
| 6. | | | $s$ | 2,5 MP |
| 7. | | $(p \rightarrow s)$ | 3-6 CP |
| 8. | $((q \rightarrow s) \rightarrow (p \rightarrow s))$ | 2-7 CP |

# Direct conditional proofs: Exercise

(115) **Modus Ponens**

$$\begin{array}{ll} & (p \to q) \\ & p \\ \hline \therefore & q \end{array}$$

(116) **Modus Tollens**

$$\begin{array}{ll} & (p \to q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(117) **Hyp. Syll.**

$$\begin{array}{ll} & (p \to q) \\ & (q \to r) \\ \hline \therefore & (p \to r) \end{array}$$

(118) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(119) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(120) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(121) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*What is the auxiliary assumption?*

Given the premises 1.-3. we can prove $(s \to (t \to u))$ !

(122) *conditional proof*:

1. $(p \vee (t \to u))$
2. $(p \to q)$
3. $((\neg s) \vee (\neg q))$
4. | Aux
5. |
6. |
7. |
8. $(s \to (t \to u))$ 4-7 CP

# Direct conditional proofs: Exercise

(115) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(116) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(117) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(118) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(119) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(120) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(121) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*What is the auxiliary assumption?*
*What are the next steps?*

Given the premises 1.-3. we can prove $(s \rightarrow (t \rightarrow u))$ !

(122) *conditional proof*:

| | | |
|---|---|---|
| 1. | $(p \vee (t \rightarrow u))$ | |
| 2. | $(p \rightarrow q)$ | |
| 3. | $((\neg s) \vee (\neg q))$ | |
| 4. | $\quad s$ | Aux |
| 5. | | |
| 6. | | |
| 7. | $\quad (t \rightarrow u)$ | |
| 8. | $(s \rightarrow (t \rightarrow u))$ | 4-7 CP |

# Direct conditional proofs: Exercise

(115) **Modus Ponens**

$$\begin{array}{ll} & (p \to q) \\ & p \\ \hline \therefore & q \end{array}$$

(116) **Modus Tollens**

$$\begin{array}{ll} & (p \to q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(117) **Hyp. Syll.**

$$\begin{array}{ll} & (p \to q) \\ & (q \to r) \\ \hline \therefore & (p \to r) \end{array}$$

(118) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(119) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(120) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(121) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*What is the auxiliary assumption?*
*What are the next steps?*

Given the premises 1.-3. we can prove $(s \to (t \to u))$ !

(122) *conditional proof*:

| | | |
|---|---|---|
| 1. | $(p \vee (t \to u))$ | |
| 2. | $(p \to q)$ | |
| 3. | $((\neg s) \vee (\neg q))$ | |
| 4. | $\mid\ s$ | Aux |
| 5. | $\mid\ (\neg q)$ | 3,4 DS |
| 6. | $\mid$ | |
| 7. | $\mid\ (t \to u)$ | |
| 8. | $(s \to (t \to u))$ | 4-7 CP |

# Direct conditional proofs: Exercise

(115) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

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$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(117) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(118) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(119) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(120) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(121) **Addition**

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*What is the auxiliary assumption?*
*What are the next steps?*

Given the premises 1.-3. we can prove $(s \rightarrow (t \rightarrow u))$ !

(122) *conditional proof*:

| | | |
|---|---|---|
| 1. | $(p \vee (t \rightarrow u))$ | |
| 2. | $(p \rightarrow q)$ | |
| 3. | $((\neg s) \vee (\neg q))$ | |
| 4. | $\mid s$ | Aux |
| 5. | $\mid (\neg q)$ | 3,4 DS |
| 6. | $\mid (\neg p)$ | 2,5 MT |
| 7. | $\mid (t \rightarrow u)$ | |
| 8. | $(s \rightarrow (t \rightarrow u))$ | 4-7 CP |

# Direct conditional proofs: Exercise

(115) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(116) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(117) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(118) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(119) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(120) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(121) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*What is the auxiliary assumption?*
*What are the next steps?*

Given the premises 1.-3. we can prove $(s \rightarrow (t \rightarrow u))$ !

(122) *conditional proof*:

| | | |
|---|---|---|
| 1. | $(p \vee (t \rightarrow u))$ | |
| 2. | $(p \rightarrow q)$ | |
| 3. | $((\neg s) \vee (\neg q))$ | |
| 4. | \| $s$ | Aux |
| 5. | \| $(\neg q)$ | 3,4 DS |
| 6. | \| $(\neg p)$ | 2,5 MT |
| 7. | \| $(t \rightarrow u)$ | 1,6 DS |
| 8. | $(s \rightarrow (t \rightarrow u))$ | 4-7 CP |

# Indirect conditional proofs

- the types of proofs we have seen are all direct proofs and direct conditional proofs
- that means the goal of the proof has been to prove $q$ from a premise $p$, where we have started with $p$ and ended with $q$
- indirect proofs aim at contradictions
- this form of argumentation uses the logic of **reductio ad absurdum**, which we have seen earlier
  - we still start with premise $p$, but we introduce the **negation of the conclusion**, i.e. $(\neg q)$ **as an auxiliary premise**
  - then we try to derive a contradiction
  - if we derive a contradiction, then we have indirectly shown that $q$ does follow from $p$ by showing that $(\neg q)$ is not compatible with $p$
  - if we don't derive a contradiction, then we have indirectly shown that the proof is not valid after all and that $q$ does not follow from $p$
- an indirect proof is a type of conditional proof since it uses an auxiliary premise
- unlike in the conditional proofs seen earlier, the auxiliary premise here is not a part of the conclusion: rather, it is the negation of the conclusion
- also: the conclusion itself doesn't need to be in a conditional form, it could even be an atomic statement

# Indirect conditional proofs

(123) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(124) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(125) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(126) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(127) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(128) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(129) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*What is the auxiliary assumption?*

Given the premises 1.-3. we can prove $p$ !

(130) *indirect proof*:

| | | |
|---|---|---|
| 1. | $(p \vee q)$ | |
| 2. | $(q \rightarrow r)$ | |
| 3. | $(\neg r)$ | |
| 4. | | Aux |
| 5. | | |
| 6. | | |
| 7. | | |
| 8. | $p$ | 4-7 IP |

# Indirect conditional proofs

(123) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(124) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(125) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(126) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(127) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(128) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(129) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*What is the auxiliary assumption?*
*What are the next steps?*

Given the premises 1.-3. we can prove $p$ !

(130) *indirect proof*:

| | | |
|---|---|---|
| 1. | $(p \vee q)$ | |
| 2. | $(q \rightarrow r)$ | |
| 3. | $(\neg r)$ | |
| 4. | \| $(\neg p)$ | Aux |
| 5. | \| | |
| 6. | \| | |
| 7. | \| | |
| 8. | $p$ | 4-7 IP |

# Indirect conditional proofs

(123) **Modus Ponens**

$$\begin{array}{ll} & (p \to q) \\ & p \\ \hline \therefore & q \end{array}$$

(124) **Modus Tollens**

$$\begin{array}{ll} & (p \to q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(125) **Hyp. Syll.**

$$\begin{array}{ll} & (p \to q) \\ & (q \to r) \\ \hline \therefore & (p \to r) \end{array}$$

(126) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(127) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(128) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(129) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*What is the auxiliary assumption?*
*What are the next steps?*

Given the premises 1.-3. we can prove $p$ !

(130) *indirect proof*:

| | | |
|---|---|---|
| 1. | $(p \vee q)$ | |
| 2. | $(q \to r)$ | |
| 3. | $(\neg r)$ | |
| 4. | $\quad (\neg p)$ | Aux |
| 5. | $\quad q$ | 1,4 DS |
| 6. | | |
| 7. | | |
| 8. | $p$ | 4-7 IP |

# Indirect conditional proofs

(123) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(124) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(125) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(126) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(127) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(128) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(129) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*What is the auxiliary assumption?*
*What are the next steps?*

Given the premises 1.-3. we can prove $p$ !

(130) *indirect proof*:

| | | |
|---|---|---|
| 1. | $(p \vee q)$ | |
| 2. | $(q \rightarrow r)$ | |
| 3. | $(\neg r)$ | |
| 4. | \| $(\neg p)$ | Aux |
| 5. | \| $q$ | 1,4 DS |
| 6. | \| $r$ | 2,5 MP |
| 7. | \| | |
| 8. | $p$ | 4-7 IP |

# Indirect conditional proofs

(123) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(124) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(125) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(126) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(127) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(128) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(129) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*What is the auxiliary assumption?*
*What are the next steps?*

Given the premises 1.-3. we can prove $p$ !

(130) *indirect proof*:

| | | |
|---|---|---|
| 1. | $(p \vee q)$ | |
| 2. | $(q \rightarrow r)$ | |
| 3. | $(\neg r)$ | |
| 4. | $\mid (\neg p)$ | Aux |
| 5. | $\mid q$ | 1,4 DS |
| 6. | $\mid r$ | 2,5 MP |
| 7. | $\mid (r \wedge (\neg r))$ | 3,6 Conj |
| 8. | $p$ | 4-7 IP |

# Indirect conditional proofs: Exercise

(131) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(132) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(133) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(134) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(135) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(136) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(137) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*What is the auxiliary assumption?*
*What are the next steps?*

Given the premises 1.-3. we can prove $m$ !

(138) *indirect proof*:

| | | |
|---|---|---|
| 1. | $(\neg m) \rightarrow (n \wedge o)$ | |
| 2. | $(n \rightarrow p)$ | |
| 3. | $(o \rightarrow (\neg p))$ | |
| 4. | | Aux |
| 5. | | |
| 6. | | |
| 7. | | |
| 8. | | |
| 9. | | |
| 10. | | |
| 11. | $m$ | 4-10 IP |

# Indirect conditional proofs

(139) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(140) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(141) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(142) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(143) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(144) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(145) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*What is the auxiliary assumption?*

Given the premises 1.-3. we can prove $m$!

(146) *indirect proof*:

| | | |
|---|---|---|
| 1. | $(\neg m) \rightarrow (n \wedge o)$ | |
| 2. | $(n \rightarrow p)$ | |
| 3. | $(o \rightarrow (\neg p))$ | |
| 4. | | Aux |
| 5. | | |
| 6. | | |
| 7. | | |
| 8. | | |
| 9. | | |
| 10. | | |
| 11. | $m$ | 4-10 IP |

# Indirect conditional proofs

(139) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

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(141) **Hyp. Syll.**

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$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

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$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

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$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(145) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*What is the auxiliary assumption?*
*What are the next steps?*

Given the premises 1.-3. we can prove $m$!

(146) *indirect proof*:

| | | |
|---|---|---|
| 1. | $(\neg m) \rightarrow (n \wedge o)$ | |
| 2. | $(n \rightarrow p)$ | |
| 3. | $(o \rightarrow (\neg p))$ | |
| 4. | $\mid (\neg m)$ | Aux |
| 5. | $\mid$ | |
| 6. | $\mid$ | |
| 7. | $\mid$ | |
| 8. | $\mid$ | |
| 9. | $\mid$ | |
| 10. | $\mid$ | |
| 11. | $m$ | 4-10 IP |

# Indirect conditional proofs

(139) **Modus Ponens**

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$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

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$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*What is the auxiliary assumption?*
*What are the next steps?*

Given the premises 1.-3. we can prove $m$!

(146) *indirect proof:*

| | | | |
|---|---|---|---|
| 1. | $(\neg m) \rightarrow (n \wedge o)$ | | |
| 2. | $(n \rightarrow p)$ | | |
| 3. | $(o \rightarrow (\neg p))$ | | |
| 4. | | $(\neg m)$ | Aux |
| 5. | | $(n \wedge o)$ | 1,4 MP |
| 6. | | | |
| 7. | | | |
| 8. | | | |
| 9. | | | |
| 10. | | | |
| 11. | $m$ | | 4-10 IP |

# Indirect conditional proofs

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$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

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$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(142) **Dis. Syll.**

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(143) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(144) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(145) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*What is the auxiliary assumption?*
*What are the next steps?*

Given the premises 1.-3. we can prove $m$!

(146) *indirect proof*:

| | | |
|---|---|---|
| 1. | $(\neg m) \rightarrow (n \wedge o)$ | |
| 2. | $(n \rightarrow p)$ | |
| 3. | $(o \rightarrow (\neg p))$ | |
| 4. | │ $(\neg m)$ | Aux |
| 5. | │ $(n \wedge o)$ | 1,4 MP |
| 6. | │ $n$ | 5 Simpl |
| 7. | │ | |
| 8. | │ | |
| 9. | │ | |
| 10. | │ | |
| 11. | $m$ | 4-10 IP |

# Indirect conditional proofs

(139) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(140) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(141) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(142) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(143) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(144) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(145) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*What is the auxiliary assumption?*
*What are the next steps?*

Given the premises 1.-3. we can prove $m$!

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| | | |
|---|---|---|
| 1. | $(\neg m) \rightarrow (n \wedge o)$ | |
| 2. | $(n \rightarrow p)$ | |
| 3. | $(o \rightarrow (\neg p))$ | |
| 4. | │ $(\neg m)$ | Aux |
| 5. | │ $(n \wedge o)$ | 1,4 MP |
| 6. | │ $n$ | 5 Simpl |
| 7. | │ $o$ | 5 Simpl |
| 8. | │ | |
| 9. | │ | |
| 10. | │ | |
| 11. | $m$ | 4-10 IP |

# Indirect conditional proofs

(139) **Modus Ponens**

$$\begin{array}{ll} & (p \to q) \\ & p \\ \hline \therefore & q \end{array}$$

(140) **Modus Tollens**

$$\begin{array}{ll} & (p \to q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(141) **Hyp. Syll.**

$$\begin{array}{ll} & (p \to q) \\ & (q \to r) \\ \hline \therefore & (p \to r) \end{array}$$

(142) **Dis. Syll.**

$$\begin{array}{ll} & (p \lor q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(143) **Simplification**

$$\begin{array}{ll} & (p \land q) \\ \hline \therefore & p \end{array}$$

(144) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \land q) \end{array}$$

(145) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \lor q) \end{array}$$

*What is the auxiliary assumption?*
*What are the next steps?*

Given the premises 1.-3. we can prove $m$!

(146) *indirect proof*:

| | | |
|---|---|---|
| 1. | $(\neg m) \to (n \land o)$ | |
| 2. | $(n \to p)$ | |
| 3. | $(o \to (\neg p))$ | |
| 4. | $\mid (\neg m)$ | Aux |
| 5. | $\mid (n \land o)$ | 1,4 MP |
| 6. | $\mid n$ | 5 Simpl |
| 7. | $\mid o$ | 5 Simpl |
| 8. | $\mid p$ | 2,6 MP |
| 9. | $\mid$ | |
| 10. | $\mid$ | |
| 11. | $m$ | 4-10 IP |

# Indirect conditional proofs

(139) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(140) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(141) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(142) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(143) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(144) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(145) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

*What is the auxiliary assumption?*
*What are the next steps?*

Given the premises 1.-3. we can prove $m$!

(146) *indirect proof*:

| | | |
|---|---|---|
| 1. | $(\neg m) \rightarrow (n \wedge o)$ | |
| 2. | $(n \rightarrow p)$ | |
| 3. | $(o \rightarrow (\neg p))$ | |
| 4. | $\mid (\neg m)$ | Aux |
| 5. | $\mid (n \wedge o)$ | 1,4 MP |
| 6. | $\mid n$ | 5 Simpl |
| 7. | $\mid o$ | 5 Simpl |
| 8. | $\mid p$ | 2,6 MP |
| 9. | $\mid (\neg p)$ | 3,7 MP |
| 10. | $\mid$ | |
| 11. | $m$ | 4-10 IP |

# Indirect conditional proofs

(139) **Modus Ponens**

$$\begin{array}{ll} & (p \to q) \\ & p \\ \hline \therefore & q \end{array}$$

(140) **Modus Tollens**

$$\begin{array}{ll} & (p \to q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(141) **Hyp. Syll.**

$$\begin{array}{ll} & (p \to q) \\ & (q \to r) \\ \hline \therefore & (p \to r) \end{array}$$

(142) **Dis. Syll.**

$$\begin{array}{ll} & (p \lor q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(143) **Simplification**

$$\begin{array}{ll} & (p \land q) \\ \hline \therefore & p \end{array}$$

(144) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \land q) \end{array}$$

(145) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \lor q) \end{array}$$

*What is the auxiliary assumption?*
*What are the next steps?*

Given the premises 1.-3. we can prove $m$!

(146) *indirect proof*:

| | | |
|---|---|---|
| 1. | $(\neg m) \to (n \land o)$ | |
| 2. | $(n \to p)$ | |
| 3. | $(o \to (\neg p))$ | |
| 4. | │ $(\neg m)$ | Aux |
| 5. | │ $(n \land o)$ | 1,4 MP |
| 6. | │ $n$ | 5 Simpl |
| 7. | │ $o$ | 5 Simpl |
| 8. | │ $p$ | 2,6 MP |
| 9. | │ $(\neg p)$ | 3,7 MP |
| 10. | │ $(p \land (\neg p))$ | 8,9 Conj |
| 11. | $m$ | 4-10 IP |