Formale Grundlagen (Logik)
Modul 04-006-1001

More on Relations

Leipzig University

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(Slides by Imke Driemel & Sandhya Sundaresan,
based on Partee, ter Meulen und Wall 1990
"Mathematical Methods in Linguistics")

A question from the last session

- Can a Cartesian Product be formed with more than two sets?
- not with our definition, we have used so far:

(1) $A \times B =_{def} \{\langle x, y\rangle \mid x \in A \text{ and } y \in B\}$

(2) $\{a, b\} \times \{1, 2\} = \left\{ \begin{array}{l} \langle a, 1\rangle, \langle a, 2\rangle, \\ \langle b, 1\rangle, \langle b, 2\rangle \end{array} \right\}$

- it is, however, easy to come with a definition for three sets:

(3) $A \times B \times C =_{def} \{\langle x, y, z\rangle \mid x \in A \text{ and } y \in B \text{ and } z \in C\}$

(4) $\{a, b\} \times \{1, 2\} \times \{\text{🐧}, \text{🐕}\} = \left\{ \begin{array}{l} \langle a, 1, \text{🐧}\rangle, \langle a, 1, \text{🐕}\rangle, \langle a, 2, \text{🐧}\rangle, \langle a, 2, \text{🐕}\rangle, \\ \langle b, 1, \text{🐧}\rangle, \langle b, 1, \text{🐕}\rangle, \langle b, 2, \text{🐧}\rangle, \langle b, 2, \text{🐕}\rangle \end{array} \right\}$

- a generalized definition would look like this:

(5) $X_1 \times \cdots \times X_n =_{def} \{\langle x_1, \ldots, x_n\rangle \mid x_i \in X_i \text{ for every } i \in \{1, \ldots, n\}\}$

Recap: Functions

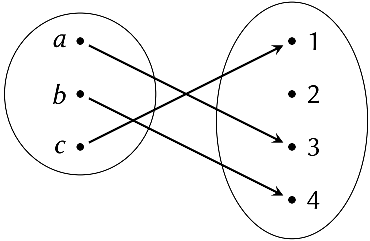
- for a relation R from A to B to count as a total function, two conditions must simultaneously hold:
  1. each element in the domain of R is paired with only one element in the range
  2. the domain of R is equal to A

(6) a. $A = \{a, b, c\}$

b. $B = \{1, 2, 3, 4\}$

(7) $Q = \{\langle a, 3\rangle, \langle b, 4\rangle, \langle c, 1\rangle\}$

(8)



Recap: Functions

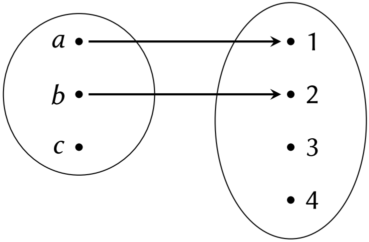
- for a relation R from A to B to count as a total function, two conditions must simultaneously hold:
  1. each element in the domain of R is paired with only one element in the range
  2. the domain of R is equal to A
- partial functions do not satify the second condition

(9) a. $A = \{a, b, c\}$

b. $B = \{1, 2, 3, 4\}$

(10) $S = \{\langle a, 1\rangle, \langle b, 2\rangle\}$

(11)



Recap: Functions

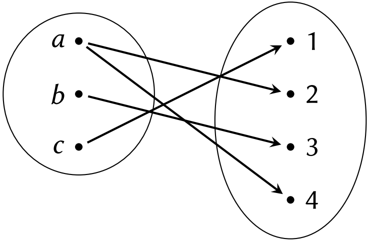
- for a relation R from A to B to count as a total function, two conditions must simultaneously hold:
  1. each element in the domain of R is paired with only one element in the range
  2. the domain of R is equal to A
- if the first condition is not satisifed, a relation is not a function

(12) a. $A = \{a, b, c\}$

b. $B = \{1, 2, 3, 4\}$

(13) $T = \{\langle a, 2\rangle, \langle b, 3\rangle, \langle a, 4\rangle, \langle c, 1\rangle\}$

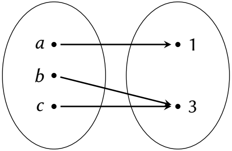
(14)



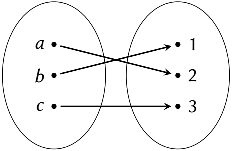
Recap: Functions

- functions from A to B are in general said to be <u>into</u> B (also called into functions) if the range of the function is a subset of B
- if the range of a function equals B, then the function is said to be <u>onto</u> B (also called onto functions or surjective functions)
- a function from A to B is called <u>one-to-one</u> (injective) iff no member of B gets mapped to by more than one member of A
- a function which is both one-to-one and onto is called a one-to-one correspondence (or bijective function)

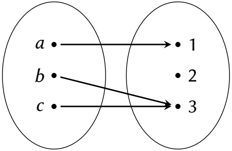
F (onto, many-to-one)



Q (onto, one-to-one)



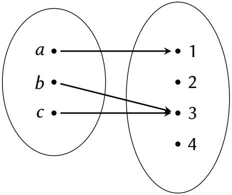
G (into, many-to-one)



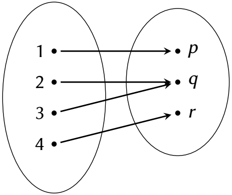
Recap: Functions

- given two functions $F : A \to B$ and $G : B \to C$, we can form a new function from A to C, the **composite** of F and G, written as $G \circ F$

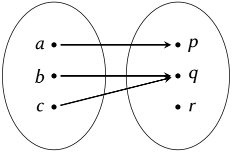
F:



G:

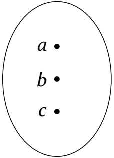


$G \circ F$:

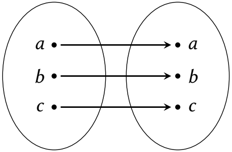


- the identity function is a function that maps each element of a set to itself: $F : A \to A$, written as id_A

A:



id_A:



Recap: Relations

- given a set A and a relation R in A ($R \subseteq A \times A$), R is **reflexive** iff for every x in A there is an ordered pair of the form $\langle x, x\rangle$ in R
  - a relation that is not reflexive is called **non-reflexive**
  - a relation which contains no ordered pair of the form $\langle x, x\rangle$ is **irreflexive**
- "is taller than" = irreflexive, "is equal to" = reflexive, "is financial supporter of" = non-reflexive
- given a set A and a relation R in A ($R \subseteq A \times A$), R is **symmetric** iff for every ordered pair $\langle x, y\rangle$ in R, the pair $\langle y, x\rangle$ is also in R
  - a relation that is not symmetric is called **non-symmetric**
  - a relation in which it is never the case that for an ordered pair $\langle x, y\rangle$, $\langle y, x\rangle$ is also a member, is **asymmetric**
  - a relation is **anti-symmetric** if whenever both $\langle x, y\rangle$ and $\langle y, x\rangle$ are in R, then $x = y$
- "self-employed" = symmetric, anti-symmetric, "friend of" = non-symmetric, "father of" = asymmetric, and "cousin of" = symmetric

Transitivity

- given a set A and a relation R in A ($R \subseteq A \times A$), R is **transitive** iff for all ordered pairs $\langle x, y \rangle$ and $\langle y, z \rangle$ in R, the pair $\langle x, z \rangle$ is also in R
  - a relation that is not transitive is called **non-transitive**
  - a relation is **intransitive** if for no pairs $\langle x, y \rangle$ and $\langle y, z \rangle$ in R, the ordered pair $\langle x, z \rangle$ is in R

(15) $A = \{1, 5, 27\}$

a. $R_1 = \{\langle 1, 5 \rangle, \langle 5, 1 \rangle, \langle 1, 1 \rangle\}$

b. $R_2 = \{\langle 5, 5 \rangle\}$

c. $R_3 = \{\langle 1, 5 \rangle, \langle 5, 27 \rangle, \langle 1, 27 \rangle, \langle 5, 1 \rangle, \langle 1, 1 \rangle, \langle 5, 5 \rangle\}$

d. $R_4 = \{\langle 1, 27 \rangle, \langle 27, 5 \rangle, \langle 1, 5 \rangle, \langle 27, 27 \rangle\}$

e. $R_5 = \{\langle 1, 27 \rangle, \langle 27, 5 \rangle, \langle 5, 1 \rangle\}$

- R_1 is non-transitive since the transitive relation for $\langle 5, 1 \rangle$ and $\langle 1, 5 \rangle$ is missing (which would be $\langle 5, 5 \rangle$)
- R_2 is transitive
- R_3 and R_4 are both transitive because for all ordered pairs $\langle x, y \rangle$ and $\langle y, z \rangle$, there is also $\langle x, z \rangle$
- R_5 is intransitive: even though there are ordered pairs of the form $\langle x, y \rangle$ and $\langle y, z \rangle$, it does not contain pairs of the form $\langle x, z \rangle$

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- what about the relations (in the set of human beings) “mother of”, “older than”, and “like” (the verb)?

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- "mother of" = intransitive,

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- "mother of" = intransitive, "older than" = transitive

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- what about the relations (in the set of human beings) "mother of", "older than", and "like" (the verb)?
- "mother of" = intransitive, "older than" = transitive, "like" = non-transitive

Connectedness

- given a set A and a relation R in A ($R \subseteq A \times A$), R is **connected** or connex iff for every two distinct elements x and y in A, the pair $\langle x, y \rangle \in R$ or $\langle y, x \rangle \in R$ or both

(16) $A = \{5, 6, 9\}$

a. $R_1 = \{\langle 5, 6 \rangle, \langle 9, 5 \rangle, \langle 6, 9 \rangle, \langle 6, 6 \rangle\}$

b. $R_2 = \{\langle 5, 5 \rangle, \langle 6, 6 \rangle, \langle 9, 9 \rangle\}$

c. $R_3 = \{\langle 6, 5 \rangle, \langle 9, 6 \rangle\}$

- R_1 is connected because all distinct pairs in A (5 and 6, 6 and 9, 5 and 9) are represented as ordered pairs in R_1
- R_2 is non-connected because none of the distinct pairs in A are represented in R_2 as distinct members of an ordered pair
- R_3 is also non-connected: an ordered pair consisting of the members 5 and 9 is missing

Connectedness

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- what about the relations "father of", "bigger than" (defined on individuals), "greater than" (defined on $\mathbb{N}$), and "same hair color as"?

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- "father of" = not connected

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- “father of” = not connected, “bigger than” = not connected

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- what about the relations "father of", "bigger than" (defined on individuals), "greater than" (defined on $\mathbb{N}$), and "same hair color as"?
- "father of" = not connected, "bigger than" = not connected, "greater than" = connected, "same hair color as" = not connected

Properties of R^{-1} and R'

- recall that the inverse of a relation R (= R^{-1}) is simply R with the members inside each ordered pair reversed
- and that the complement of a relation R (= R') contains all the ordered pairs (in the Cartesian Product of which R is a subset) that are not in R
- certain properties are preserved from R to R^{-1} and R'

R (not $\varnothing$)	R^{-1}	R'
reflexive	reflexive	irreflexive
irreflexive	irreflexive	reflexive
symmetric	symmetric	symmetric
asymmetric	asymmetric	non-symmetric
antisymmetric	antisymmetric	depends on R
transitive	transitive	depends on R
intransitive	intransitive	depends on R
connected	connected	depends on R

Properties of R^{-1} and R'

- let us have a closer look at the reflexivity properties

R (not $\varnothing$)	R^{-1}	R'
reflexive	reflexive	irreflexive
irreflexive	irreflexive	reflexive
...	...	...

- by definition, a reflexive relation R contains all pairs of the form $\langle x, x\rangle$
- since R^{-1} has all pairs of R but with the order reversed, every pair $\langle x, x\rangle$ will also be in R^{-1}
- if R is reflexive, it contains all pairs $\langle x, x\rangle$, hence there are no pairs $\langle x, x\rangle$ left to be in R'
- this necessarily makes R' irreflexive
- the same logic can be applied to the second row

Properties of R^{-1} and R'

- now let us consider symmetry properties

R (not $\varnothing$)	R^{-1}	R'
...	...	...
symmetric	symmetric	symmetric
asymmetric	asymmetric	non-symmetric

- by definition, a relation R is asymmetric if it contains pairs of the form $\langle x, y\rangle$, but not their respective counterparts $\langle y, x\rangle$
- if R contains the pairs $\langle x, y\rangle$, the inverse R^{-1} contains the pairs $\langle y, x\rangle$ instead of the pairs $\langle x, y\rangle$, hence it is also asymmetric
- R', however, still contains symmetric pairs, hence it is non-symmetric

(17) a. $A = \{a, b, c\}$

b. $R = \{\langle a, b\rangle, \langle a, c\rangle\}$

(18) $R' = \{\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle b, a\rangle, \langle c, a\rangle, \langle b, c\rangle, \langle c, b\rangle\}$

- which ordered pairs make R' non-symmetric?

Properties of R^{-1} and R'

- now let us consider symmetry properties

R (not $\varnothing$)	R^{-1}	R'
...	...	...
symmetric	symmetric	symmetric
asymmetric	asymmetric	non-symmetric

- by definition, a relation R is symmetric if for every pair of the form $\langle x, y \rangle$, there exists also the counterpart $\langle y, x \rangle$
- the inverse R^{-1} simply reverses the order of elements within the pairs $\langle y, x \rangle$ and $\langle x, y \rangle$
- so we can conclude that if R is symmetric then $R^{-1} = R$
- but why is R' also symmetric?

Properties of R^{-1} and R'

- now let us consider symmetry properties

R (not $\varnothing$)	R^{-1}	R'
...	...	...
symmetric	symmetric	symmetric
asymmetric	asymmetric	non-symmetric

- why is R' also symmetric?
- because the opposite assumption leads to an absurd conclusion!
  1. assumption: R is symmetric and R' is non-symmetric
  2. R' contains pairs of the form $\langle x, y\rangle$ but not their $\langle y, x\rangle$ counterparts
  3. if pair $\langle y, x\rangle$ is not in R', then it must be in $(R')'$ which is R
  4. but since R is symmetric, R must also contain $\langle x, y\rangle$
  5. this leads to an absurd conclusion: $\langle x, y\rangle$ is in R but also in R'
  6. thus we conclude that our initial assumption is false
  7. if R' cannot be non-symmetric, it must be symmetric
- this mode of reasoning is also called a *reductio ad absurdum* proof in logic (proof by contradiction)

Equivalence relations and classes

- an **equivalence relation** is one which is **reflexive**, **symmetric**, and **transitive**
- "=" is the most typical equivalence relation; others are: "same height as" or "same age as "
- for every equivalence relation there is a natural way to divide the set for which it is defined into mutally exclusive (disjoint) subsets, called **equivalence classes**
- an equivalence class $[x]$ is a set of all elements that are related to x by some equivalence relation

(19) $[x] = \{y \mid \langle x, y \rangle \in R\}$, where R is an equivalence relation

- every pair of equivalence classes (i.e., a pair of sets) is disjoint (has no shared members)
- example: given a set A, R is an equivalence relation (why?)

(20) a. $A = \{book, hook, bat, hat\}$

b. $R \subseteq A \times A = \{\langle book, book \rangle, \langle hook, hook \rangle, \langle hat, hat \rangle, \langle bat, bat \rangle, \langle book, hook \rangle, \langle hook, book \rangle, \langle hat, bat \rangle, \langle bat, hat \rangle\}$

c. 2 equivalence classes defined by R: $\{book, hook\}$ and $\{bat, hat\}$

- what is the equivalence relation R from (20) in natural language? "rhymes with"

Partitions

- there is a close correspondence between equivalence classes and partitions
- given a non-empty set A, a **partition** of A is a collection of non-empty subsets of A such that
  1. for any two distinct subsets X and Y, $X \cap Y = \varnothing$
  2. the union of all the subsets equals A
- members of a partition are called the **cells** of the partition

(21) Let $A = \{a, b, c, d, e\}$. Then $P = \{\{a, c\}, \{b, e\}, \{d\}\}$ is a partition of A, because:

$\rightsquigarrow$ every cell of P is disjoint, i.e.
$\{a, c\} \cap \{b, e\} = \varnothing; \{b, e\} \cap \{d\} = \varnothing; \{a, c\} \cap \{d\} = \varnothing$

$\rightsquigarrow$ the big union of the cells equals A, i.e. $\{a, c\} \cup \{b, e\} \cup \{d\} = \{a, b, c, d, e\}$

- are the following sets partitions of A?

(22)
a. $P_1 = \{\{a, b, c\}, \{d, e\}\}$ *yes*
b. $P_2 = \{\{a, b, c\}, \{e\}\}$ *no*
c. $P_3 = \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}\}$ *yes*
d. $P_4 = \{\{a, b, c\}, \{d, e, c\}\}$ *no*
e. $P_5 = \{\{a, b, c, d, e\}\}$ *yes*

Partitions and equivalence relations

- there is a close correspondence between equivalence classes and partitions

(23) Given a partition of set A,
the relation $R = \{\langle x, y\rangle \mid x \text{ and } y \text{ are in the same cell of the partition}\}$
is an equivalence relation.

(24) Given an equivalence relation R in A,
there exists a partition of A in which x and y are in the same cell iff
x and y are related by R

- equivalence classes specified by R are simply the cells of the partition
- an equivalence relation in A is sometimes said to *induce a partition of A*
- here is an example going from equivalence relation to partition

(25) a. $A = \{1, 2, 3, 4\}$

b. $R = \{\langle 1, 1\rangle, \langle 2, 2\rangle, \langle 1, 2\rangle, \langle 2, 1\rangle, \langle 3, 3\rangle, \langle 4, 4\rangle, \langle 3, 4\rangle, \langle 4, 3\rangle\}$

c. 2 equivalence classes defined by R: $\{1, 2\}$ and $\{3, 4\}$

d. partition on A, induced by R: $P_R = \{\{1, 2\}, \{3, 4\}\}$

Partitions and equivalence relations

- there is a close correspondence between equivalence classes and partitions

(26) Given a partition of set A,
the relation $R = \{\langle x, y\rangle \mid x$ and y are in the same cell of the partition$\}$
is an equivalence relation.

(27) Given an equivalence relation R in A,
there exists a partition of A in which x and y are in the same cell iff
x and y are related by R

- equivalence classes specified by R are simply the cells of the partition
- an equivalence relation in A is sometimes said to *induce a partition of A*
- we can also go from a partition to the equivalence relation

(28) a. $B = \{1, 2, 3, 4, 5\}$

b. partition on B, induced by R: $Q_R = \{\{1, 2\}, \{3, 5\}, \{4\}\}$

c. 3 equivalence classes defined by R: $\{1, 2\}$ and $\{3, 5\}$ and $\{4\}$

d. $R = \{\langle 1, 1\rangle, \langle 2, 2\rangle, \langle 1, 2\rangle, \langle 2, 1\rangle, \langle 3, 3\rangle, \langle 5, 5\rangle, \langle 3, 5\rangle, \langle 5, 3\rangle, \langle 4, 4\rangle\}$

Exercise

- provide equivalence relation, equivalence classes and the partition

(29) a. $X = \{France, Ghana, Belgium, Ecuador, Brazil\}$

b. $R =$ is on the same continent as

(30) a. partition on X, induced by R:
$P_R = \{\{France, Belgium\}, \{Ecuador, Brazil\}, \{Ghana\}\}$

b. 3 equivalence classes defined by R:
$\{France, Belgium\}$ and $\{Ecuador, Brazil\}$ and $\{Ghana\}$

c. $R = \{\langle France, France\rangle, \langle Belgium, Belgium\rangle, \langle France, Belgium\rangle, \langle Belgium, France\rangle, \langle Ecuador, Ecuador\rangle, \langle Brazil, Brazil\rangle, \langle Ecuador, Brazil\rangle, \langle Brazil, Ecuador\rangle, \langle Ghana, Ghana\rangle\}$

Orderings

- an ordering is a binary relation which is **transitive** and additionally:

weak order	strong order
reflexive	irreflexive
anti-symmetric	asymmetric

- which of the following relations on set A are orderings? if so, are they strong or weak orderings?

(31) $A = \{a, b, c, d\}$

(32) a. $R_1 = \{\langle a, b\rangle, \langle a, c\rangle, \langle a, d\rangle, \langle b, c\rangle, \langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle d, d\rangle\}$ *weak*

b. $R_2 = \{\langle b, a\rangle, \langle c, b\rangle, \langle c, a\rangle\}$ *strong*

c. $R_3 = \{\langle a, b\rangle, \langle a, d\rangle, \langle b, c\rangle, \langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle d, d\rangle\}$ *not an order*

d. $R_4 = \{\langle d, c\rangle, \langle d, b\rangle, \langle d, a\rangle, \langle c, b\rangle, \langle c, a\rangle, \langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle d, d\rangle, \langle b, a\rangle\}$

e. $R_5 = \{\langle a, b\rangle, \langle a, c\rangle, \langle a, d\rangle, \langle b, c\rangle\}$

f. $R_6 = \{\langle b, a\rangle, \langle b, b\rangle, \langle a, a\rangle, \langle c, c\rangle, \langle d, d\rangle, \langle c, b\rangle, \langle c, a\rangle\}$

g. $R_7 = \{\langle d, c\rangle, \langle d, b\rangle, \langle d, a\rangle, \langle c, b\rangle, \langle c, a\rangle, \langle b, a\rangle\}$

h. $R_8 = \{\langle a, b\rangle, \langle a, c\rangle, \langle a, d\rangle, \langle b, c\rangle, \langle d, a\rangle\}$