Formale Grundlagen (Logik)
Modul 04-006-1001

Relations & Functions

Leipzig University

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(Slides by Imke Driemel & Sandhya Sundaresan,
based on Partee, ter Meulen und Wall 1990
"Mathematical Methods in Linguistics")

Recap: Set theoretic equalities

- last time we looked at general laws about set equivalence relations
    1. Idempotent Laws
    2. Commutative Laws
    3. Associative Laws
    4. Distributive Laws
    5. Identity Laws
    6. Complement Laws
    7. DeMorgan's Laws
    8. Consistency Principle

- these laws help us deal with complex expressions over sets

Recap: Set theoretic equalities

(1) **Idempotent Laws**:

a. $X \cup X = X$

b. $X \cap X = X$

(2) **Commutative Laws**:

a. $X \cup Y = Y \cup X$

b. $X \cap Y = Y \cap X$

(3) **Associative Laws**:

a. $(X \cup Y) \cup Z = X \cup (Y \cup Z)$

b. $(X \cap Y) \cap Z = X \cap (Y \cap Z)$

(4) **Identity Laws**:

a. $X \cup \varnothing = X$

b. $X \cap \varnothing = \varnothing$

c. $X \cup U = U$

d. $X \cap U = X$

(5) **Distributive Laws**:

a. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

b. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(6) **Complement Laws**:

a. $A \cup A' = U$

b. $(A')' = A$

c. $A \cap A' = \varnothing$

d. $A - B = A \cap B'$

(7) **DeMorgan's Laws**:

a. $(A \cup B)' = A' \cap B'$

b. $(A \cap B)' = A' \cup B'$

(8) **Consistency Principle**:

a. $X \subseteq Y$ iff $X \cup Y = Y$

b. $X \subseteq Y$ iff $X \cap Y = X$

Recap: Relations

- an (ordered) **tuple** is a finite, ordered collection of members

(9) a. $X = \langle a, b, c, d \rangle$

b. $Y = \langle d, c, b, a \rangle$

c. $X \neq Y$

(10) a. $X = \langle b, b, b \rangle$

b. $Y = \langle b \rangle$

c. $X \neq Y$

- important subtype of tuples: tuples which consist of exactly two members: **ordered pairs**
- we can derive tuples, in particular ordered pairs, from sets by forming the **Cartesian Product** of these sets

(11) $A \times B =_{def} \{\langle x, y \rangle \mid x \in A \text{ and } y \in B\}$

(12) $\{a, b\} \times \{1, 2\} = \left\{ \begin{array}{l} \langle a, 1 \rangle, \langle a, 2 \rangle, \\ \langle b, 1 \rangle, \langle b, 2 \rangle \end{array} \right\}$

- ordered pairs can be understood as instances of a relation
- relations form subsets of sets created by the Cartesian Product

(13) $R \subseteq A \times B$

Recap: Relations

- the smallest sets A and B such that $R \subseteq A \times B$ (for some given R) are $A = \{a \mid \langle a, b \rangle \in R \text{ for some } b\}$ and $B = \{b \mid \langle a, b \rangle \in R \text{ for some } a\}$
- A and B are the projections onto R's first and second coordinate
- A = **domain** of R; B = the **range** of R
- the **set complement** R' contains all the ordered pairs in the Cartesian product of A and B which are not members of the relation R

(14) $R' =_{def} (A \times B) - R$

- the **inverse of a relation R**, written as R^{-1}, is the set containing all the ordered pairs in R, but with the first and second member in each ordered pair reversed

Functions

- a function is a special kind of relation
- functions are central to our study of meaning in natural language – also called natural language semantics
- for a relation R from A to B to count as a total function, two conditions must simultaneously hold:
  1. each element in the domain of R is paired with only one element in the range
  2. the domain of R is equal to A
- in other words:
  1. every member of A is used at most once as the first coordinate of the ordered pairs in R
  2. every member of A is used at least once as the first coordinate of the ordered pairs in R
- the first condition is what makes R a function, the second condition is what makes the function total

Functions

- for a relation R from A to B to count as a total function, two conditions must simultaneously hold:
  1. each element in the domain of R is paired with only one element in the range
  2. the domain of R is equal to A
- the relations P, Q, R from sets A to B are total functions because every member in A is used exactly once

(15) a. $A = \{a, b, c\}$

b. $B = \{1, 2, 3, 4\}$

(16) a. $P = \{\langle a, 1\rangle, \langle b, 2\rangle, \langle c, 3\rangle\}$

b. $Q = \{\langle a, 3\rangle, \langle b, 4\rangle, \langle c, 1\rangle\}$

c. $R = \{\langle a, 3\rangle, \langle b, 2\rangle, \langle c, 2\rangle\}$

Functions

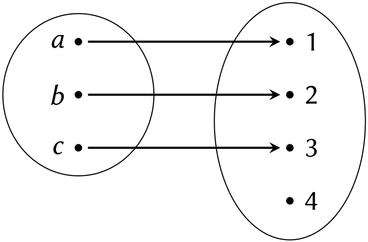
- for a relation R from A to B to count as a total function, two conditions must simultaneously hold:
  1. each element in the domain of R is paired with only one element in the range
  2. the domain of R is equal to A
- the relations P from sets A to B is a total function because every member in A is used exactly once

(17) a. $A = \{a, b, c\}$

b. $B = \{1, 2, 3, 4\}$

(18) $P = \{\langle a, 1\rangle, \langle b, 2\rangle, \langle c, 3\rangle\}$

(19)



Functions

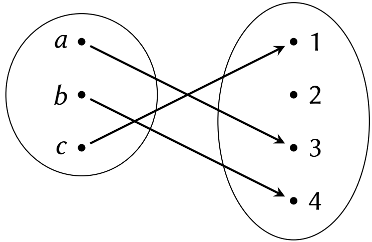
- for a relation R from A to B to count as a total function, two conditions must simultaneously hold:
  1. each element in the domain of R is paired with only one element in the range
  2. the domain of R is equal to A
- the relation Q from sets A to B is a total function because every member in A is used exactly once

(20) a. $A = \{a, b, c\}$

b. $B = \{1, 2, 3, 4\}$

(21) $Q = \{\langle a, 3\rangle, \langle b, 4\rangle, \langle c, 1\rangle\}$

(22)



Functions

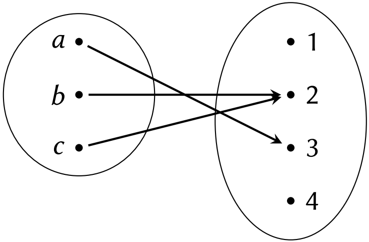
- for a relation R from A to B to count as a total function, two conditions must simultaneously hold:
  1. each element in the domain of R is paired with only one element in the range
  2. the domain of R is equal to A
- the relation R from sets A to B is a total function because every member in A is used exactly once

(23) a. $A = \{a, b, c\}$

b. $B = \{1, 2, 3, 4\}$

(24) $R = \{\langle a, 3\rangle, \langle b, 2\rangle, \langle c, 2\rangle\}$

(25)



Functions

- for a relation R from A to B to count as a total function, two conditions must simultaneously hold:
    1. each element in the domain of R is paired with only one element in the range
    2. the domain of R is equal to A
- the relations S, T, V from sets A to B are <u>NOT</u> total functions – why?

(26) a. $A = \{a, b, c\}$

b. $B = \{1, 2, 3, 4\}$

(27) a. $S = \{\langle a, 1\rangle, \langle b, 2\rangle\}$

b. $T = \{\langle a, 2\rangle, \langle b, 3\rangle, \langle a, 4\rangle, \langle c, 1\rangle\}$

c. $V = \{\langle a, 2\rangle, \langle a, 3\rangle, \langle b, 4\rangle\}$

Functions

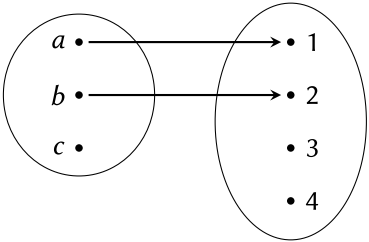
- for a relation R from A to B to count as a total function, two conditions must simultaneously hold:
  1. each element in the domain of R is paired with only one element in the range
  2. the domain of R is equal to A
- the relation S from sets A to B is not a total function because not every member in A is used

(28) a. $A = \{a, b, c\}$

b. $B = \{1, 2, 3, 4\}$

(29) $S = \{\langle a, 1\rangle, \langle b, 2\rangle\}$

(30)



Functions

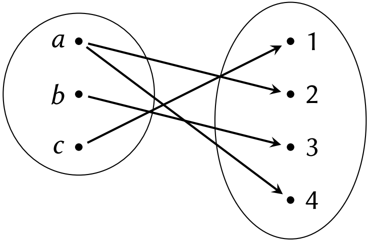
- for a relation R from A to B to count as a total function, two conditions must simultaneously hold:
  1. each element in the domain of R is paired with only one element in the range
  2. the domain of R is equal to A
- the relation T from sets A to B is not a total function (not even a partial one, see below) because the element a is related to both 2 and 4

(31) a. $A = \{a, b, c\}$

b. $B = \{1, 2, 3, 4\}$

(32) $T = \{\langle a, 2\rangle, \langle b, 3\rangle, \langle a, 4\rangle, \langle c, 1\rangle\}$

(33)



Functions

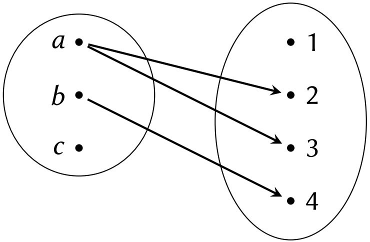
- for a relation R from A to B to count as a total function, two conditions must simultaneously hold:
    1. each element in the domain of R is paired with only one element in the range
    2. the domain of R is equal to A
- the relation V from sets A to B is not a (total) function because not every member in A is used and a relates to both 2 and 3

(34) a. $A = \{a, b, c\}$

b. $B = \{1, 2, 3, 4\}$

(35) $V = \{\langle a, 2\rangle, \langle a, 3\rangle, \langle b, 4\rangle\}$

(36)



Function talk, set talk

- a function that is a subset of $A \times B$ is called a function *from A to B*, also written as $\boxed{F : A \to B}$
- a function that is a subset of $A \times A$ is called a function *in A*
- members of the domain of a function are called its **arguments**
- members of the range of a function are called its **values**
- functions can also be presented more dynamically:

(37) $F(a) = 2$ *read: F maps a onto 2*

- regularly used in mathematics where the pairing of argument and value is often specified by operations such as division and addition, etc.
- we will stick with the more static set-theoretic talk here

(38) a. $F(x) = x + 1$

b. $F = \{\langle x, y \rangle \mid y = x + 1\}$

Partial functions

- the conditions for functionhood we gave above identified **total functions**
  1. each element in the domain of R is paired with only one element in the range
  2. the domain of R is equal to A
- a **partial function** is a function that does not satisfy the second condition
  1. each element in the domain of R is paired with only one element in the range
  2. ~~the domain of R is equal to A~~
- recall the relations S, T, V from sets A to B – which one is a partial function?

(39) a. $A = \{a, b, c\}$

b. $B = \{1, 2, 3, 4\}$

(40) a. $S = \{\langle a, 1\rangle, \langle b, 2\rangle\}$

b. $T = \{\langle a, 2\rangle, \langle b, 3\rangle, \langle a, 4\rangle, \langle c, 1\rangle\}$

c. $V = \{\langle a, 2\rangle, \langle a, 3\rangle, \langle b, 4\rangle\}$

Partial functions

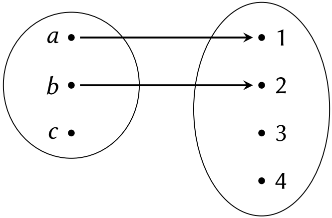
- the conditions for function we gave above identified **total functions**
  1. each element in the domain of R is paired with only one element in the range
  2. the domain of R is equal to A
- a **partial function** is a function that does not satisfy the second condition
  1. each element in the domain of R is paired with only one element in the range
  2. ~~the domain of R is equal to A~~
- S is a partial function!

(41) a. $A = \{a, b, c\}$

b. $B = \{1, 2, 3, 4\}$

(42) $S = \{\langle a, 1\rangle, \langle b, 2\rangle\}$

(43)



Exercise

- say why or why not the following relations are
  - not functions
  - total functions
  - partial functions

(44) a. $F = \{\langle x, y\rangle \mid y = \textit{mother of } x\}$

b. $G = \{\langle x, y\rangle \mid y = \textit{brother of } x\}$

c. $Q = \{\langle x, y\rangle \mid y = \textit{twin of } x\}$

Exercise

- say why or why not the following relations are
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b. $G = \{\langle x, y \rangle \mid y = \textit{brother of } x\}$

c. $Q = \{\langle x, y \rangle \mid y = \textit{twin of } x\}$

- F is a function since no one has more than one (biological) mother; F is a total function since everybody has a (biological) mother

Exercise

- say why or why not the following relations are
  - not functions
  - total functions
  - partial functions

(44) a. $F = \{\langle x, y\rangle \mid y = \textit{mother of } x\}$

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c. $Q = \{\langle x, y\rangle \mid y = \textit{twin of } x\}$

- F is a total function since everybody has a (biological) mother; F is a function since no one has more than one (biological) mother
- G is not a function since a person can have more than one brother

Exercise

- say why or why not the following relations are
  - not functions
  - total functions
  - partial functions

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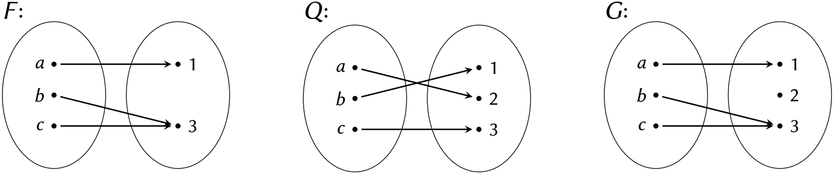
c. $Q = \{\langle x, y\rangle \mid y = \textit{twin of } x\}$

- F is a total function since everybody has a (biological) mother; F is a function since no one has more than one (biological) mother
- G is not a function since a person can have more than one brother
- Q is a partial function since not everyone has a twin sibling; Q is a function since there can only be one twin for each x

More function talk

- it is sometimes useful to state whether the range of a function from A to B is equal to the set B
- functions from A to B are in general said to be *into* B (also called into functions) if the range of the function is a subset of B
- if the range of a function equals B, then the function is said to be *onto* B (also called onto functions or surjective functions)
- which of the illustrated functions are into? which are onto?

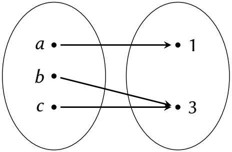


- F and Q are onto (surjective) functions, G is not
- F, Q, G are into functions

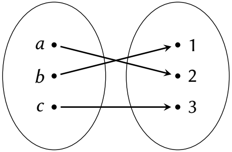
More function talk

- A function from A to B is called *one-to-one* (injective) iff no member of B gets mapped to by more than one member of A
- A function that is not one-to-one is sometimes called *many-to-one*
- a function which is both one-to-one and onto is called a one-to-one correspondence (surjective + injective = bijective function)
- how can we apply these terms to the following functions?

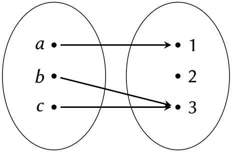
F:



Q:



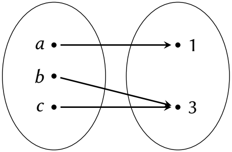
G:



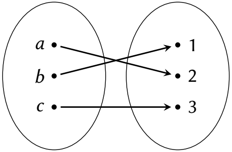
More function talk

- A function from A to B is called *one-to-one* (injective) iff no member of B gets mapped to by more than one member of A
- A function that is not one-to-one is sometimes called *many-to-one*
- a function which is both one-to-one and onto is called a one-to-one correspondence (surjective + injective = bijective function)
- how can we apply these terms to the following functions?

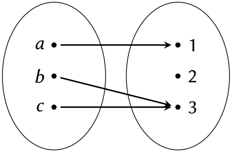
F:



Q:



G:

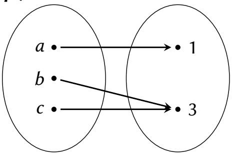


- F and G are many-to-one functions

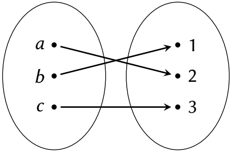
More function talk

- A function from A to B is called *one-to-one* (injective) iff no member of B gets mapped to by more than one member of A
- A function that is not one-to-one is sometimes called *many-to-one*
- a function which is both one-to-one and onto is called a one-to-one correspondence (surjective + injective = bijective function)
- how can we apply these terms to the following functions?

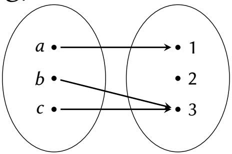
F:



Q:



G:



- F and G are many-to-one functions
- Q is a one-to-one correspondence

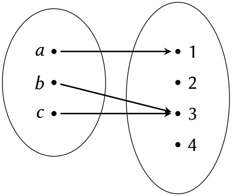
Function composition

- we can combine two functions to create a third function
- given two functions $F : A \to B$ and $G : B \to C$, we can form a new function from A to C.
- this new function is called the **composite** of F and G, written as $G \circ F$

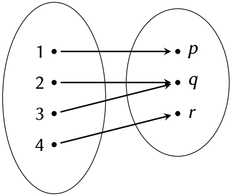
(45) $G \circ F =_{def} \{\langle x, z\rangle \mid \text{for some } y, \langle x, y\rangle \in F \text{ and } \langle y, z\rangle \in G\}$

- below we illustrate with an example:

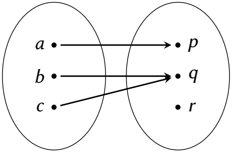
F:



G:



$G \circ F$:



- $G \circ F$ only produces sets of ordered pairs if the range of F is not disjoint from the domain of G ($range(F) \cap domain(G) \neq \emptyset$)

Exercise

- provide the composites:

(46) a. $S = \{\langle a, 5\rangle, \langle f, 6\rangle, \langle g, 7\rangle\}$

b. $T = \{\langle 6, Marge\rangle, \langle 7, Homer\rangle, \langle 5, Lisa\rangle\}$

c. $T \circ S =$

(47) a. $W = \{\langle English, C\rangle, \langle German, R\rangle, \langle French, V\rangle\}$

b. $X = \{\langle C, Marge\rangle, \langle Q, Homer\rangle, \langle R, Lisa\rangle, \langle V, Bart\rangle\}$

c. $X \circ W =$

Exercise

- provide the composites:

(46) a. $S = \{\langle a, 5\rangle, \langle f, 6\rangle, \langle g, 7\rangle\}$

b. $T = \{\langle 6, Marge\rangle, \langle 7, Homer\rangle, \langle 5, Lisa\rangle\}$

c. $T \circ S = \{\langle a, Lisa\rangle, \langle f, Marge\rangle, \langle g, Homer\rangle\}$

(47) a. $W = \{\langle English, C\rangle, \langle German, R\rangle, \langle French, V\rangle\}$

b. $X = \{\langle C, Marge\rangle, \langle Q, Homer\rangle, \langle R, Lisa\rangle, \langle V, Bart\rangle\}$

c. $X \circ W =$

Exercise

- provide the composites:

(46) a. $S = \{\langle a, 5\rangle, \langle f, 6\rangle, \langle g, 7\rangle\}$

b. $T = \{\langle 6, \textit{Marge}\rangle, \langle 7, \textit{Homer}\rangle, \langle 5, \textit{Lisa}\rangle\}$

c. $T \circ S = \{\langle a, \textit{Lisa}\rangle, \langle f, \textit{Marge}\rangle, \langle g, \textit{Homer}\rangle\}$

(47) a. $W = \{\langle \textit{English}, C\rangle, \langle \textit{German}, R\rangle, \langle \textit{French}, V\rangle\}$

b. $X = \{\langle C, \textit{Marge}\rangle, \langle Q, \textit{Homer}\rangle, \langle R, \textit{Lisa}\rangle, \langle V, \textit{Bart}\rangle\}$

c. $X \circ W = \{\langle \textit{German}, \textit{Lisa}\rangle, \langle \textit{English}, \textit{Marge}\rangle, \langle \textit{French}, \textit{Bart}\rangle\}$

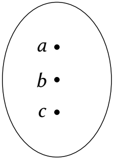
Identity functions

- the identity function is a function that maps each element of a set to itself: $F : A \rightarrow A$, written as id_A

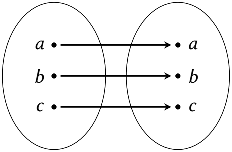
(48) $F =_{def} \{\langle x, x\rangle \mid x \in A\}$

- we illustrate again with an example:

A:



id_A:



- for a function $F : A \rightarrow B$, the following holds:

(49) a. $F \circ id_A = F$

b. $id_B \circ F = F$

Back to relations

- we will now discuss the following properties of binary relations
  1. reflexivity
  2. symmetry
- these properties apply to relations *in* a set, e.g. in $A \times A$, not to relations from A to B, where $B \neq A$

Reflexivity

- given a set A and a relation R in A, R is **reflexive** iff for every x in A there is an ordered pair of the form $\langle x, x \rangle$ in R
  - a relation that is not reflexive is called **non-reflexive**
  - a relation which contains no ordered pair of the form $\langle x, x \rangle$ is **irreflexive**
- every irreflexive relation is non-reflexive, but not every non-reflexive relation is irreflexive

(50) $A = \{1, 2, 3\}$

a. $R_1 = \{\langle 1, 1 \rangle, \langle 3, 3 \rangle, \langle 3, 2 \rangle, \langle 1, 2 \rangle\}$

b. $R_2 = \{\langle 1, 1 \rangle, \langle 2, 2 \rangle, \langle 3, 3 \rangle\}$

c. $R_3 = \{\langle 1, 1 \rangle, \langle 2, 2 \rangle, \langle 3, 3 \rangle, \langle 3, 2 \rangle, \langle 1, 2 \rangle, \langle 3, 1 \rangle\}$

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Reflexivity

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- “is taller than” = irreflexive, “is equal to” = reflexive, “is a financial supporter of” = non-reflexive

Symmetry

- given a set A and a relation R in A, R is **symmetric** iff for every ordered pair $\langle x, y \rangle$ in R, the pair $\langle y, x \rangle$ is also in R
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a. $R_1 = \{\langle a, b \rangle, \langle b, a \rangle\}$

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- R_4 is asymmetric because none of the members has a corresponding reverse
- R_5 is non-symmetric (the pair $\langle a, c\rangle$ doesn't have a reverse) and also anti-symmetric (the only pair for which there is a reverse is of the reflexive type: $\langle b, b\rangle$)

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- "is self-employed by" = symmetric, anti-symmetric, "is friend of" = non-symmetric, "is father of" = asymmetric, and "is cousin of" = symmetric