Formale Grundlagen (Logik)
Modul 04-006-1001

Set theory III: ordered sets, relations

Leipzig University

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(Slides by Imke Driemel & Sandhya Sundaresan,
based on Partee, ter Meulen und Wall 1990
"Mathematical Methods in Linguistics")

Recap: Empty set, set membership, subsets

- the empty set is a set with no members, formally notated as: $\varnothing$ (or: $\{\}$)
- the empty set is a subset of every set:

(1) $\varnothing \subseteq Y$, for any set Y

- this follows from our definition of subsets:

(2) For any two sets X and Y, if every member of X is a member of Y, then X is a subset of Y

- for X <u>not</u> to be a subset of Y, X must have at least one member that is not in Y
- but the empty set by definition has no members
- hence, the empty set is a subset of every set

Recap: Empty set, set membership, subsets

- member of a set $\neq$ subset of a set
- consider the following sets:

(3) a. $A = \{Mercury, Venus, Earth, Mars, Jupiter\}$

b. $B = \{Mars\}$

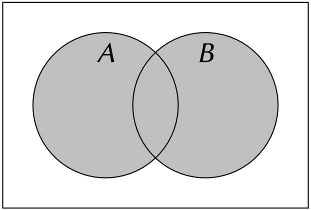
c. $X = \{Mercury, Venus, Earth, \{Mars\}, Jupiter\}$

- *Mars* is a member of *A*: $Mars \in A$
- *Mars* is not a subset of *A*: $Mars \not\subseteq A$ (*Mars* is not a set)
- *B* is a subset of *A*: $B \subseteq A$ (every member in *B*, which is only *Mars*, is also in *A*)
- *B* is a member of *X*: $B \in X$
- *B* is a not a subset of *X*: $B \not\subseteq X$ (*Mars* is a member of *B* but not *X*)
- $\{\{Mars\}\}$ is a subset of *X*: $\{\{Mars\}\} \subseteq X$

Recap: set operations

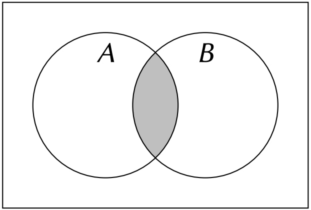
set union:

(4) $A \cup B =_{def}$
$\{x \mid \text{x} \in A \text{ or } \text{x} \in B\}$

(5) 

set intersection:

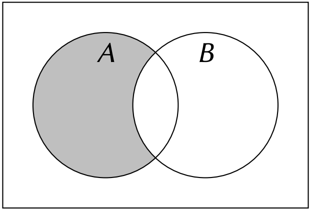
(6) $A \cap B =_{def}$
$\{x \mid \text{x} \in A \text{ and } \text{x} \in B\}$

(7) 

Recap: set operations

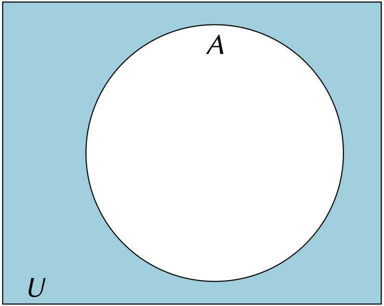
set difference:

(8) $A - B =_{def}$ $\{x \mid x \in A \text{ and } x \notin B\}$

(9) 

set complement:

(10) $A' =_{def} \{x \mid x \notin A\}$

(11) 

Set operations: exercise

- provide the newly formed sets:

(12) a. $Y = \{y \mid \text{y is yellow}\}$

b. $Y' =$

(13) a. $Q = \{\alpha, \gamma, 100, 3, 4, 5\}$, $M = \{\alpha, \gamma, \beta, 102, \mathit{Hulk}, 5\}$

b. $Q - M =$

c. $M - Q =$

(14) a. $X = \varnothing$, $G = \{a, b, c\}$

b. $\wp(G) =$

c. $G - X =$

d. $X - G =$

Set operations: exercise

- provide the newly formed sets:

(12) a. $Y = \{y \mid \text{y is yellow}\}$

b. $Y' = \{y \mid \text{y is not yellow}\}$

(13) a. $Q = \{\alpha, \gamma, 100, 3, 4, 5\}$, $M = \{\alpha, \gamma, \beta, 102, \mathit{Hulk}, 5\}$

b. $Q - M = \{100, 3, 4\}$

c. $M - Q = \{\beta, 102, \mathit{Hulk}\}$

(14) a. $X = \varnothing$, $G = \{a, b, c\}$

b. $\wp(G) = \{\{a, b, c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a\}, \{b\}, \{c\}, \varnothing\}$

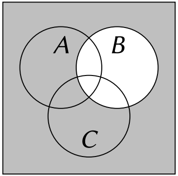
c. $G - X = \{a, b, c\} = G$

d. $X - G = \varnothing$

Set operations: exercise

- provide the set relations for the Venn diagrams

(15)



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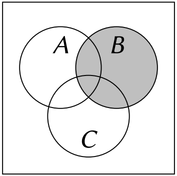

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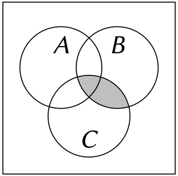
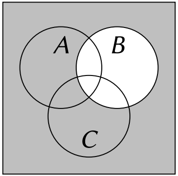


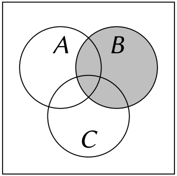
Set operations: exercise

- provide the set relations for the Venn diagrams

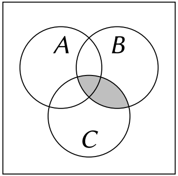
(15) B'



(16) B



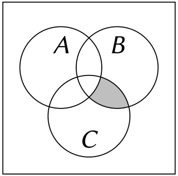
(17) $B \cap C$



Set operations: exercise

- provide the set relations for the Venn diagrams

(18)



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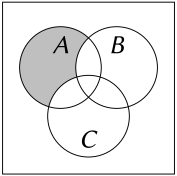

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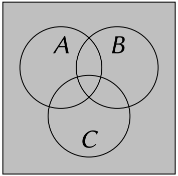
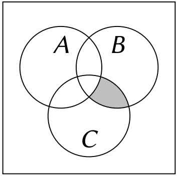


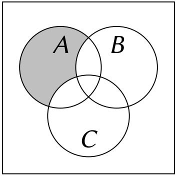
Set operations: exercise

- provide the set relations for the Venn diagrams

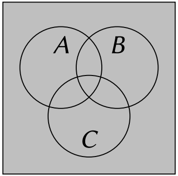
(18) $(B \cap C) - A$



(19) $A - (B \cup C)$



(20) U



Set theoretic equalities

- there are a number of general laws pertaining to sets which follow from the definitions of union, intersection etc.
- these laws help us deal with complex expressions over sets
- most of these are quite intuitive, or will be once we draw Venn Diagrams for them
  1. Idempotent Laws
  2. Commutative Laws
  3. Associative Laws
  4. Distributive Laws
  5. Identity Laws
  6. Complement Laws
  7. DeMorgan's Laws (Augustus De Morgan, 1806–1871, English mathematician)
  8. Consistency Principle

Set theoretic equalities

- **Idempotent Laws**: everything which is in X *or* in X simply amounts to everything which is in X, similarly for everything which is in X *and* in X

(21) a. $X \cup X = X$

b. $X \cap X = X$

- **Commutative Laws**: everything which is in X *or* in Y (or both) is the same as everything which is in Y *or* in X (or both); same goes for intersection

(22) a. $X \cup Y = Y \cup X$

b. $X \cap Y = Y \cap X$

Set theoretic equalities

- **Associative Laws**: the order in which we combine three sets by the operation of union does not matter, and the same is true if the operation is intersection

(23) a. $(X \cup Y) \cup Z = X \cup (Y \cup Z)$

b. $(X \cap Y) \cap Z = X \cap (Y \cap Z)$

- **Identity Laws**: evident from the definitions of union, intersection, the empty set, and the universal set

(24) a. $X \cup \varnothing = X$

b. $X \cap \varnothing = \varnothing$

(25) a. $X \cup U = U$

b. $X \cap U = X$

Set theoretic equalities

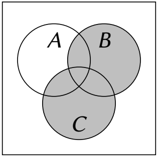
- **Distributive Laws**:

(26) a. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

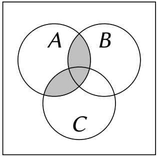
b. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

- we illustrate with Venn diagrams for $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

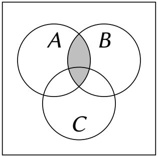
(27) $B \cup C$



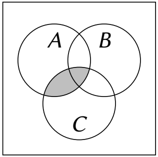
(28) $A \cap (B \cup C)$



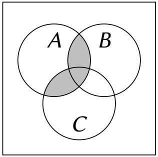
(29) $A \cap B$



(30) $A \cap C$



(31) $(A \cap B) \cup (A \cap C)$



Set theoretic equalities

- **Complement Laws**:

(32) a. $A \cup A' = U$

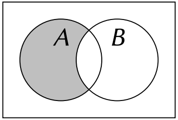
b. $(A')' = A$

(33) a. $A \cap A' = \varnothing$

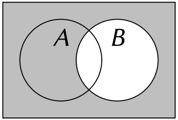
b. $A - B = A \cap B'$

- we illustrate with Venn diagrams for $A - B = A \cap B'$

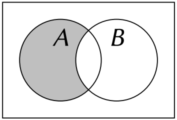
(34) $A - B$



(35) B'



(36) $A \cap B'$



Set theoretic equalities

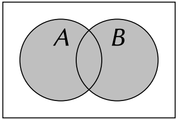
- **DeMorgan's Laws**:

(37) a. $(A \cup B)' = A' \cap B'$

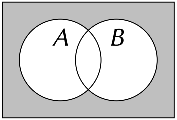
b. $(A \cap B)' = A' \cup B'$

- we illustrate with Venn diagrams for $(A \cup B)' = A' \cap B'$

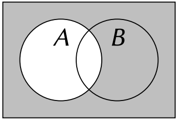
(38) $A \cup B$



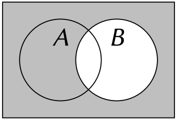
(39) $(A \cup B)'$



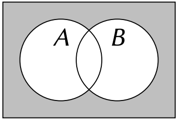
(40) A'



(41) B'



(42) $A' \cap B'$



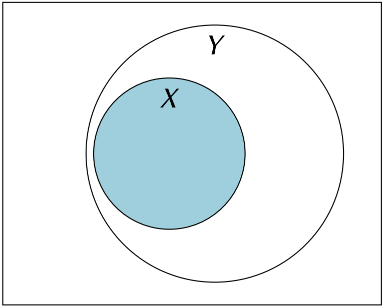
Set theoretic equalities

- **Consistency Principle**: concerned with the mutual consistency of the definitions of union, intersection, and subset (iff = if and only if)

(43) a. $X \subseteq Y$ iff $X \cup Y = Y$

b. $X \subseteq Y$ iff $X \cap Y = X$

(44)



Set theoretic equalities

- why are we thinking about set theoretic equalities?
- because we can simplify set-theoretic complex expressions by applying these laws; here is an example:

(45) $(A \cup B) \cup (B \cap C)' = U$

(46) $(A \cup B) \cup (B \cap C)'$

$$
\begin{aligned}
&= (A \cup B) \cup (B' \cup C') && \textit{DeMorgan}_{(X \cap Y)' = X' \cup Y'} \\
&= A \cup (B \cup (B' \cup C')) && \textit{Associative}_{(X \cup Y) \cup Z = X \cup (Y \cup Z)} \\
&= A \cup ((B \cup B') \cup C') && \textit{Associative}_{(X \cup Y) \cup Z = X \cup (Y \cup Z)} \\
&= A \cup (U \cup C') && \textit{Complement}_{X \cup X' = U} \\
&= A \cup (C' \cup U) && \textit{Commutative}_{X \cup Y = Y \cup X} \\
&= A \cup U && \textit{Identity}_{X' \cup U = U} \\
&= U && \textit{Identity}_{X \cup U = U}
\end{aligned}
$$

From sets to tuples

- members of a set are unordered

(47) a. $A = \{a, b, c, d\}$

b. $B = \{d, c, b, a\}$

c. $A = B$

- we will now introduce a new type of mathematical object, where the order of the members does matter: the (ordered) **tuple**
- an (ordered) tuple is a finite, ordered collection of members
- the members of an ordered tuple are written in pointy brackets: $\langle\ \rangle$

(48) a. $X = \langle a, b, c, d\rangle$

b. $Y = \langle d, c, b, a\rangle$

c. $X \neq Y$

(49) a. $X = \langle b, b, b\rangle$

b. $Y = \langle b\rangle$

c. $X \neq Y$

From sets to tuples

- we will mostly be concerned with a subtype of tuples, i.e. tuples which consist of exactly two members: **ordered pairs**, example:

(50) $X = \langle a, b \rangle$

- we can derive tuples, in particular ordered pairs, from sets
- the **Cartesian Product** of sets A and B, written as $A \times B$, is the set consisting of all ordered pairs formed from the sets A and B, where the first element of the pair is taken from A and the second element from B

(51) $A \times B =_{def} \{\langle x, y \rangle \mid x \in A \text{ and } y \in B\}$

(52) $\{a, b\} \times \{1, 2\} = \left\{ \begin{array}{l} \langle a, 1\rangle, \langle a, 2\rangle, \\ \langle b, 1\rangle, \langle b, 2\rangle \end{array} \right\}$

- the Cartesian Product creates a set; the members of this set, e.g. $\langle a, 1\rangle$, $\langle a, 2\rangle$, etc., are not ordered relative to each other; but the members of each ordered pair are ordered

From sets to tuples: exercise

- the **Cartesian Product** of sets A and B, written as $A \times B$, is the set consisting of all ordered pairs formed from the sets A and B, where the first element of the pair is taken from A and the second element from B

(53) $A \times B =_{def} \{\langle x, y\rangle \mid x \in A \text{ and } y \in B\}$

(54) $\{a, b\} \times \{1, 2\} = \left\{ \begin{array}{l} \langle a, 1\rangle, \langle a, 2\rangle, \\ \langle b, 1\rangle, \langle b, 2\rangle \end{array} \right\}$

- create the following sets:

(55) $X = \{Homer, Marge\}$; $Y = \{Bart, Lisa\}$

a. $X \times Y = \{\langle Homer, Bart\rangle, \langle Homer, Lisa\rangle, \langle Marge, Bart\rangle, \langle Marge, Lisa\rangle\}$

b. $Y \times X = \{\langle Bart, Homer\rangle, \langle Bart, Marge\rangle, \langle Lisa, Homer\rangle, \langle Lisa, Marge\rangle\}$

c. $Y \times Y = \{\langle Bart, Bart\rangle, \langle Bart, Lisa\rangle, \langle Lisa, Bart\rangle, \langle Lisa, Lisa\rangle\}$

- relations are built from Cartesian Products, ordered pairs can express instances of a relation
- what is the relation described by $X \times Y$ in (55)? what is the relation described by $Y \times X$?
- $X \times Y$ expresses the relation "parent (of)"; $Y \times X$ expresses the relation "child (of)"

Relations

- informally, a relation expresses a connection between elements of two sets
- example: “mother of” is a relation that holds between the elements of two sets of people, specifically mothers and their children
- in natural language, a relationship is expressed by a linguistic object that we call **predicate**
- example: the transitive verb “eat” denotes a relation between elements of two sets of objects, the eaters and the eatees

Relations

- relations form subsets of sets created by the Cartesian Product

(56) $R \subseteq A \times B$

- the smallest sets A and B such that $R \subseteq A \times B$ (for some given R) are
  $A = \{a \mid \langle a, b \rangle \in R \text{ for some } b\}$ and
  $B = \{b \mid \langle a, b \rangle \in R \text{ for some } a\}$
- these two sets are called the projections of R onto the first and second coordinate, respectively
- the projection of R onto the first coordinate is called the **domain** of R
- the projection of R onto the second coordinate is called the **range** of R

Relations

- we will again look at a concrete example:

(57) a. $X = \{Homer, Bart, Marge\}$

b. $Y = \{Bart, Lisa\}$

(58) Cartesian Product:
$X \times Y = \{\langle Homer, Bart\rangle, \langle Homer, Lisa\rangle, \langle Bart, Bart\rangle,$
$\langle Bart, Lisa\rangle, \langle Marge, Bart\rangle, \langle Marge, Lisa\rangle\}$

- now we define relation “parent of” as a set $C \subseteq X \times Y$ such that: in each ordered pair in C, the first member is a parent of the second

(59) a. $C = \{\langle a, b\rangle \mid a \text{ parent of } b\}$

b. $C = \{\langle Homer, Bart\rangle, \langle Homer, Lisa\rangle, \langle Marge, Bart\rangle, \langle Marge, Lisa\rangle\}$

(60) a. domain of $C = \{Homer, Marge\}$

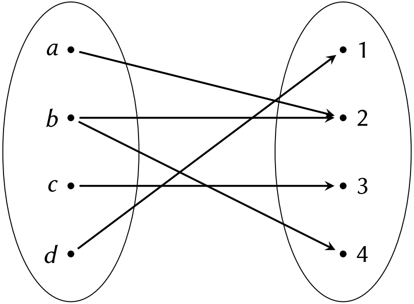
b. range of $C = \{Bart, Lisa\}$

Relations

- relations can also be illustrated with diagrams such as the one below

(61) $\{\langle a, 2\rangle, \langle b, 2\rangle, \langle b, 4\rangle, \langle c, 3\rangle, \langle d, 1\rangle\}$

(62)



Relations

- since relations are sets, we can also provide the **set complement of a relation**
- let us come back to our original example:

(63) a. $X = \{Homer, Bart, Marge\}$

b. $Y = \{Bart, Lisa\}$

(64) a. $X \times Y = \{\langle Homer, Bart\rangle, \langle Homer, Lisa\rangle, \langle Bart, Bart\rangle, \langle Bart, Lisa\rangle, \langle Marge, Bart\rangle, \langle Marge, Lisa\rangle\}$

b. $C = \{\langle Homer, Bart\rangle, \langle Homer, Lisa\rangle, \langle Marge, Bart\rangle, \langle Marge, Lisa\rangle\}$

- the set complement for some relation $R \subseteq A \times B$ is formally given below:

(65) $R' =_{def} (A \times B) - R$

- the set complement R' contains all the ordered pairs in the Cartesian product of A and B which are not members of the relation R
- for our set C above we can now form C'

(66) $C' = \{\langle Bart, Bart\rangle, \langle Bart, Lisa\rangle\}$

Relations

- we can also form the **inverse of a relation**
- again we look at our original example:

(67) a. $X = \{Homer, Bart, Marge\}$

b. $Y = \{Bart, Lisa\}$

(68) a. $X \times Y = \{\langle Homer, Bart\rangle, \langle Homer, Lisa\rangle, \langle Bart, Bart\rangle, \langle Bart, Lisa\rangle, \langle Marge, Bart\rangle, \langle Marge, Lisa\rangle\}$

b. $C = \{\langle Homer, Bart\rangle, \langle Homer, Lisa\rangle, \langle Marge, Bart\rangle, \langle Marge, Lisa\rangle\}$

- the inverse of a relation R, written as R^{-1}, is the set containing all the ordered pairs in R, but with the first and second member in each ordered pair reversed
- for our set C above we can now form C^{-1}

(69) $C^{-1} = \{\langle Bart, Homer\rangle, \langle Lisa, Homer\rangle, \langle Bart, Marge\rangle, \langle Lisa, Marge\rangle\}$

Relations: exercise

- given the sets X and Y:

(70) a. $X = \{2, 4, 1\}$

b. $Y = \{3, 2\}$

- form the Cartesian Product $X \times Y$
- define the relation "greater than" as a set $C \subseteq X \times Y$ such that: in each ordered pair in C, the first member is greater then the second; also provide C
- what is the domain and the range of the relation C?
- form C' and C^{-1}

(71) Cartesian Product:
$X \times Y = \{\langle 2, 3\rangle, \langle 2, 2\rangle, \langle 4, 3\rangle, \langle 4, 2\rangle, \langle 1, 3\rangle, \langle 1, 2\rangle\}$

(72) a. $C = \{\langle a, b\rangle \mid a \text{ is greater than } b\}$

b. $C = \{\langle 4, 3\rangle, \langle 4, 2\rangle\}$

(73) a. domain of C $= \{4\}$

b. range of C $= \{3, 2\}$

(74) a. $C' = \{\langle 2, 3\rangle, \langle 2, 2\rangle, \langle 1, 3\rangle, \langle 1, 2\rangle\}$

b. $C^{-1} = \{\langle 3, 4\rangle, \langle 2, 4\rangle\}$