Formale Grundlagen (Logik)
Modul 04-006-1001

Set Theory II: set operations

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(Slides by Imke Driemel & Sandhya Sundaresan,
based on Partee, ter Meulen und Wall 1990
“Mathematical Methods in Linguistics”)

Recap: Set Theory

Notion of a set

A set is a collection of distinct, unordered, potentially unbounded, abstract objects.

- distinct: repetition of a member does not change the set, i.e. $A = B$ in (1)

(1) $A = \{x, x, x, x, x\}; B = \{x\}$

- unordered: relative order of the set members does not matter, i.e. $X = Y$ in (2)

(2) $X = \{a, b, c, d\}; Y = \{d, a, b, c\}$

- potentially unbounded: sets can contain a non-finite number of elements

(3) $C = \{x \mid \text{x is a natural number}\}$

- abstract: sets can contain any kind of object

- we learned about different notations

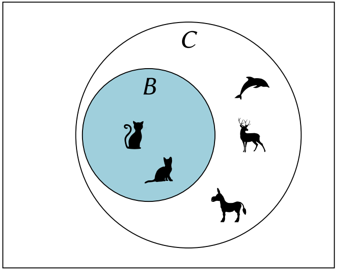
(4) a. $B = \{$  $, \ldots \}$ *list notation*

b. $C = \{$  $, \ldots \}$

(5) a. $B = \{x \mid \text{x is a cat}\}$ *predicate notation*

b. $C = \{y \mid \text{y is an animal}\}$

(6) *Venn diagram*



Recap: Set Theory

- we learned about supersets, subsets, and proper subsets

(7) For any two sets X and Y, if every member of X is a member of Y, then:

a. X is a subset of Y, and

b. Y is a superset of X

(8) A set X is a proper subset of a set Y if:

a. every member of X is also a member of Y, but

b. not every member of Y is a member of X (i.e. Y has at least one member that is not in X)

The empty set

- the empty set is a set with no members, notated as: $\varnothing$ (or: $\{\}$)
- the following holds:

(9) $\varnothing \subseteq Y$, for any set Y

- in prose: *the empty set is a subset of every set*
- here is why:
  1. we said a set X is a subset of set Y if every member of X is also a member of Y
  2. this means that for X <u>not</u> to be a subset of Y, X must have at least one member that is not in Y
  3. but the empty set by definition has no members
  4. so it is impossible for the empty set to have a member that is not also a member of some set Y, and hence, it is impossible for the empty set not to be a subset of Y

Set member vs. subset

- members of sets and subsets of sets both express part-whole relationships
- but a member of a set $\neq$ the subset of a set
- consider the following sets:

(10) a. $A = \{Mercury, Venus, Earth, Mars, Jupiter\}$

b. $B = \{Mars\}$

  - *Mars* is a member of *A*; we write: $Mars \in A$ ($\in$ is read as "element of")
  - but *Mars* is not a set; so: $Mars \not\subseteq A$
  - but set *B* is a subset of *A*; so: $\{Mars\} \subseteq A$
  - every member of *B* (in this case the single member *Mars*) is also a member of *A*

- now consider set *Y*

(11) $Y = \{Mercury, Venus, Earth, \{Mars\}, Jupiter\}$

  - $\{Mars\}$ is a member of *Y*; so: $\{Mars\} \in Y$
  - is $\{Mars\}$ a subset of *Y*?
  - recall that a set *X* is a subset of set *Y* if every member of *X* is also a member of *Y*

Set member vs. subset

- now consider set Y

(11) $Y = \{Mercury, Venus, Earth, \{Mars\}, Jupiter\}$

  - $\{Mars\}$ is a member of Y; so: $\{Mars\} \in Y$
  - is $\{Mars\}$ a subset of Y?
  - recall that a set X is a subset of set Y if every member of X is also a member of Y
  - $\{Mars\}$ is not a subset of Y; i.e. $\{Mars\} \not\subseteq Y$
  - because the single member of $\{Mars\}$, which is *Mars*, is not a member of Y
  - only $\{Mars\}$ is a member of Y

Set member vs. subset

- now consider set Y

(11) $Y = \{Mercury, Venus, Earth, \{Mars\}, Jupiter\}$

  - recall that a set X is a subset of set Y if every member of X is also a member of Y
  - $\{Mars\}$ is not a subset of Y; i.e. $\{Mars\} \not\subseteq Y$
  - because the single member of $\{Mars\}$, which is *Mars*, is not a member of Y
  - only $\{Mars\}$ is a member of Y

- determine the truth of the following statements for set A

(12) $A = \{b, \{c\}\}$

a. $b \not\subseteq \mathrm{A}$

b. $\{b\} \subseteq \mathrm{A}$

c. $\{c\} \subseteq \mathrm{A}$

d. $\{\{c\}\} \subseteq \mathrm{A}$

e. $\{b\} \in \mathrm{A}$

Set member vs. subset

- now consider set Y

(11) $Y = \{Mercury, Venus, Earth, \{Mars\}, Jupiter\}$

  - recall that a set X is a subset of set Y if every member of X is also a member of Y
  - $\{Mars\}$ is not a subset of Y; i.e. $\{Mars\} \not\subseteq Y$
  - because the single member of $\{Mars\}$, which is *Mars*, is not a member of Y
  - only $\{Mars\}$ is a member of Y

- determine the truth of the following statements for set A

(12) $A = \{b, \{c\}\}$

a. $b \not\subseteq \text{A}$ **true**

b. $\{b\} \subseteq \text{A}$ **true**

c. $\{c\} \subseteq \text{A}$ **false**

d. $\{\{c\}\} \subseteq \text{A}$ **true**

e. $\{b\} \in \text{A}$ **false**

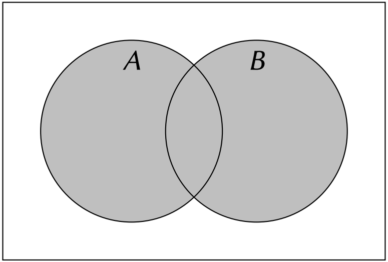
Set operations

- sets are mathematical objects, just like numbers
- you can take two numbers and perform some mathematic operations over them (e.g. addition, subtraction, division) to create a third number
- similarly, you can take two sets and perform operations over them to create a third set
- we will look at three such set operations: set union, set intersection, and set difference

Set operations: set union

- the union of two sets A and B is a new set that contains the members of A plus the members of B

(13) $A \cup B =_{def} \{x \mid \text{x} \in A \text{ or x} \in B\}$

(14) 

- the union of *A* and *B* describes elements which can be in *A* or in *B* or in both (inclusive disjunction)

Set operations: set union

- the union of two sets A and B is a new set that contains the members of A plus the members of B

(13) $A \cup B =_{def} \{x \mid \text{x} \in A \text{ or x} \in B\}$

- examples:

(15) a. $A = \{2, 4, \theta\}$

b. $B = \{14, 8,$  $\}$

c. $A \cup B = \{2, 4, \theta, 14, 8,$  $\}$

(16) a. $Y = \{y \mid \text{y is a student}\}$

b. $X = \{x \mid \text{x is happy}\}$

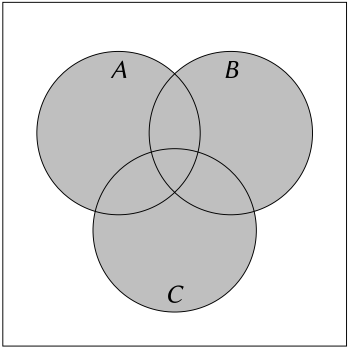
c. $X \cup Y = \{v \mid \text{v is a student or v is happy}\}$

Set operations: set union

- set theoretic union can easily be generalized to apply to more than two sets
- in this case we often write the union sign in front of the sets to be operated on

(17) $A \cup B \cup C = \bigcup\{A, B, C\}$

(18)

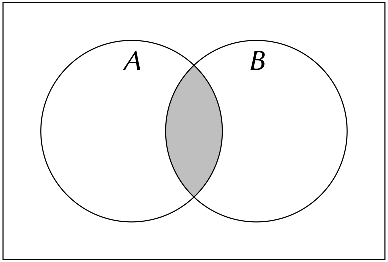


Set operations: set intersection

- the intersection of two sets A and B is a new set that contains only the members which are in both A and B

(19) $A \cap B =_{def} \{x \mid \mathrm{x} \in A \text{ and } \mathrm{x} \in B\}$

(20)



Set operations: set intersection

- the intersection of two sets A and B is a new set that contains only the members which are in both A and B

(19) $A \cap B =_{def} \{x \mid \text{x} \in A \text{ and x} \in B\}$

- examples:

(21) a. $A = \{2, 9, \heartsuit, 4, \theta\}$

b. $B = \{\heartsuit, 14, 8, 9,$  $\}$

c. $A \cap B = \{\heartsuit, 9\}$

(22) a. $Y = \{y \mid \text{y is a singer}\}$

b. $X = \{x \mid \text{x is a songwriter}\}$

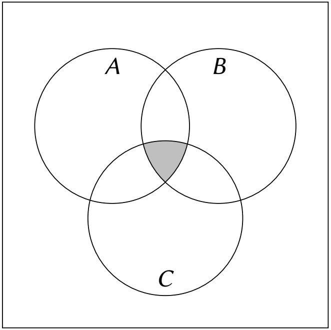
c. $X \cap Y = \{v \mid \text{v is a singer and v is a songwriter}\}$

Set operations: set intersection

- again, we can generalize set intersection to apply to more than two sets
- we have an equivalently different notation for expressing the intersection of multiple sets

(23) $A \cap B \cap C = \bigcap\{A, B, C\}$

(24)



Set operations

- the operations of set union and set intersection look very abstract at the moment
- but they are actually quite relevant for understanding certain linguistic expressions
- set union is important to understand natural language (inclusive) disjunctions
  - inclusive: *I can offer you cake or coffee.*
  - exclusive: *You can either take the day off or keep working.*
- set intersection is important to understand natural language conjunctions
  - *He is a father and husband now.*
  - *My uncle is blind and funny.*
  - not logical conjunction: *Sue and Mary are together*; *His friends and enemies agreed only on one point*
- set intersection is also important for modification
  - *a happy student*, *a French dessert*

Set operations: exercise

- provide the newly formed sets:

(25) a. $Y = \{y \mid \text{y is yellow}\}$, $C = \{c \mid \text{c is blue}\}$, $D = \{d \mid \text{d is green}\}$

b. $\bigcap\{C, D, Y\} = \{x \mid \text{x is yellow and x is blue and x is green}\}$

(26) a. $Q = \{\alpha, \gamma, 100\}$, $M = \{Hulk, Spiderman, \{Batman, Joker\}, Thor\}$

b. $Q \cup M = \{\alpha, \gamma, 100, Hulk, Spiderman, \{Batman, Joker\}, Thor\}$

c. $M \cup Q = \{\alpha, \gamma, 100, Hulk, Spiderman, \{Batman, Joker\}, Thor\}$

d. $M \cap Q = \varnothing$

(27) a. $X = \varnothing$, $G = \{a, b, c\}$

b. $G \cup X = \{a, b, c\} = G$

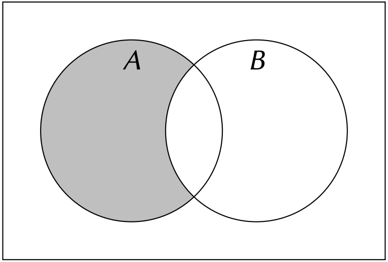
c. $G \cap X = \varnothing$

Set operations: set difference

- the difference between sets A and B ($A - B$) yields another set whose members include all the members in A that are not in B (sometimes written as $A \backslash B$)

(28) $A - B =_{def} \{x \mid \text{x} \in A \text{ and x} \notin B\}$

(29)



Set operations: set difference

- the difference between sets A and B yields another set whose members include all the members in A that are not in B (in some notations also $A \backslash B$)

(28) $A - B =_{def} \{x \mid \text{x} \in A \text{ and x} \notin B\}$

- examples:

(30) a. $A = \{2, 9, \heartsuit, 4, \theta\}$

b. $B = \{\heartsuit, 14, 8, 9,$  $\}$

c. $A - B = \{2, 4, \theta\}$

(31) a. $Y = \{y \mid \text{y is married}\}$

b. $X = \{x \mid \text{x is a man}\}$

c. $X - Y = \{v \mid \text{v is a man and v is not married}\}$

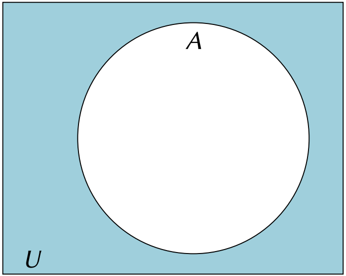
Complement of a set

- the complement of a set A is the set containing everything not in A

(32) $A' =_{def} \{x \mid x \notin A\}$

- when we talk about sets this happens against a background of assumed objects which form the **universe** of discourse for what we say
- but there are no specific rules about what exactly the members of the universe must be: this depends on the context of discussion

(33)



The universe is also a set – thus also called the universal set; written as U

U is not explicitly mentioned, but it is always understood to be there

$A' = U - A$

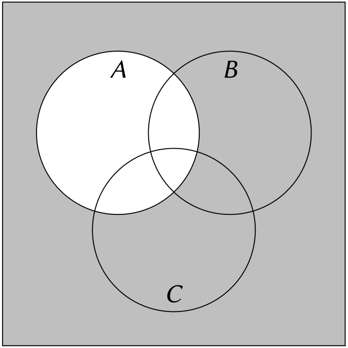
Complement of a set

- the complement of a set A is the set containing everything not in A

(32) $A' =_{def} \{x \mid \mathrm{x} \notin A\}$

- we can also formulate the complement of a set with other sets present

(34)



Power set

- sometimes we need to refer to the set whose members are all the subsets of a given set A
- this set is called the power set of A, written as $\wp(A)$

(35) a. $A = \{a, b\}$

b. $\wp(A) = \{\{a, b\}, \{a\}, \{b\}, \varnothing\}$

- the name "power set" derives from the fact that if the cardinality of A is some natural number n, then $\wp(A)$ has cardinality 2^n
- $2^n = 2$ raised to the n power, or $2 \times 2 \times 2 \times 2 \times \cdots \times 2$ (n times)
- so for our example this means:
  - A has 2 members
  - $\wp(A)$ has cardinality 2^2
  - $2 \times 2 = 4$
  - $\wp(A)$ has 4 members