Formale Grundlagen (Logik)
Modul 04-006-1001

Introduction, Set Theory I

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(Slides by Imke Driemel & Sandhya Sundaresan, based on Partee, ter Meulen und Wall 1990 "Mathematical Methods in Linguistics")

Introduction

- main goal of this course: introduction to the basics of formal logic, so that you may eventually apply the skills you learn here to understand the logical basis of natural language
- logic is a system of **truth** and **inference**:
  - determines the conditions under which a proposition ($\approx$ sentence) is true or false
  - inferring whether a given proposition is true or false given what we know about the truth or falsehood of a different proposition
- topics of this course:
  - set theory (Mengenlehre)
  - relations, functions
  - logical operators: $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$
  - truth tables (Wahrheitstafeln)
  - ways to set up proofs
- logic provides us with many of the tools to do formal semantics (meaning of natural language sentences)

Set Theory

- set theory forms a central part of logic and involves the formal study of sets
- invented by Georg Cantor (German mathematician, 1845–1918) in 1874
- what is a set?

> **Notion of a set**
>
> A set is a collection of distinct, unordered, potentially unbounded, abstract objects.

- these objects are called **members** or **elements** of the set
- the notation for a set is $\{\}$
- the members of a set are enclosed within these brackets

(1) $A = \{a, b, c, d, e, f\}$

- this is called a **list notation** of a set (because the members of the set A are simply listed)

Set Theory

- what is a set?

Notion of a set

A set is a collection of **distinct**, unordered, potentially unbounded, abstract objects.

- what does distinct mean?
- this feature points to the fact that repetition of a member does not change the set
- so the set in (2) is not a set containing five members; rather, it's exactly the same as a set containing a single member, namely x

(2) $A = \{x, x, x, x, x\} = \{x\}$

- other properties related to set-membership:
  1. an object is either a member of a set or not
  2. there is no such thing as a halfway, multiple, or gradual membership

Set Theory

- what is a set?

Notion of a set

A set is a collection of distinct, **unordered**, potentially unbounded, abstract objects.

- what does unordered mean?
- simply that: the relative order of the set members in which they are listed does not matter
- so the sets A and B in (3) are the same set, i.e. $A = B$
- they contain the same members, but with different orders

(3) $A = \{a, b, c, d\}, B = \{d, a, b, c\}$

- in this sense, sets are crucially different from another type of logical object, namely an ordered (n-)tuple

Set Theory

- what is a set?

Notion of a set

A set is a collection of distinct, unordered, **potentially unbounded**, abstract objects.

- what does potentially unbounded mean?
- there is no limit to the number of members it can have
- example: the set containing the natural numbers is a set with an infinite number of members
- another example: the students participating in this course (finite)
- we could even have a set containing no members at all: this would be the empty set, conventionally written as: $\varnothing$ (or, alternatively, as $\{\}$)

Set Theory

- what is a set?

Notion of a set

A set is a collection of distinct, unordered, potentially unbounded, **abstract** objects.

- what does abstract mean?
- set members can be literally anything!
- here are some examples of possible sets:
  - the set of all prime numbers
  - the set of all characters of the Simpsons
  - this random set: $\{1, 3, \psi, 5.3\}$
  - another random set: $\left\{ b, a, 4, x, \alpha, \right.$  $\left. \right\}$
  - a bunch of animals: {  }

Set Theory

- **recursive** sets: since a set can contain members of any kind, it can also contain other sets as members

(4) $A = \{a, y, \{2, d\}, \beta, \heartsuit, 3\}$

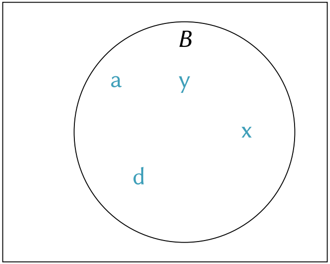
- when we talk about the members of a set, it is important to keep separate the objects/concepts vs. the names of the object
- we typically use names or descriptions to refer to objects
- when a set contains those names or descriptions, it refers to the objects that these names denote, and not to the names themselves
- if we want to refer to the names rather than the objects, we enclose the name inside single quotes (convention)

(5) $A = \{'Hulk'\} \neq \{$  $\}$ vs. $B = \{Hulk\} = \{$  $\}$

Set Theory with Venn Diagrams

- so far we have used list notations which list the members of a set from left to right
- this implies an order but we said that order does not really matter
- a more intuitive way of illustrating sets (without implying order) is via **Venn diagrams** (John Venn, English mathematician, logician, 1834–1923)

(6) $B = \{y, x, a, d\}$ *list notation*

(7) 

Venn diagram

Set Theory with predicate notation

- instead of listing the members of a set, we could simply describe the set by naming a property that all of its members share
- this is called **predicate notation**

(8) a. $A = \{Homer, Bart, Maggie, Marge, Lisa\}$ *list notation*

b. $A = \{x \mid \text{x is a Simpson}\}$ *predicate notation*

- the | should be read as "such that"
- one advantage of predicate notation over list notation is that it can be easlily used to describe sets with a very large or even infinite number of members

(9) a. $B = \{x \mid \text{x is blue}\}$

b. $C = \{x \mid \text{x is a star in the sky}\}$

c. $D = \{x \mid \text{x is related to Tom Hanks}\}$

Superset and subset

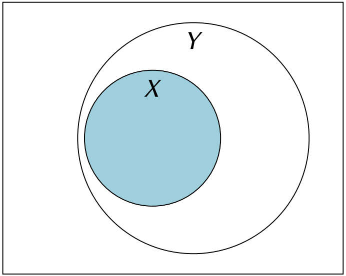
- a fundamental type of relationship between two sets is the subset-superset relationship

(10) For any two sets X and Y, if every member of X is a member of Y, then:

a. X is a subset of Y, and

b. Y is a superset of X

(11)



subset relation is notated with $\subseteq$:
$X \subseteq Y$

superset relation is notated with $\supseteq$:
$Y \supseteq X$

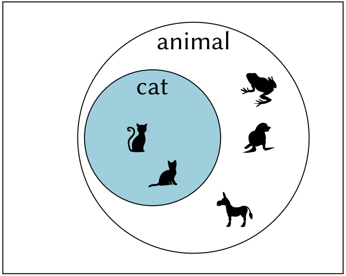
Superset and subset

- an example:

(12) a. $B = \{x \mid \text{x is a cat}\}$

b. $C = \{y \mid \text{y is an animal}\}$

(13)



every cat is also an animal . . .

. . . thus, every member of B (the set of cats) is also a member of C (the set of animals)

$B \subseteq C$ and $C \supseteq B$

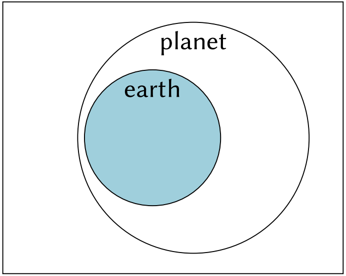
Superset and subset

- another example:

(14) a. $D = \{\text{earth}\}$

b. $E = \{y \mid \text{y is a planet}\}$

(15)



earth is a planet ...

... thus, the only member of D (earth) is also a member of E (the set of planets)

$D \subseteq E$ and $E \supseteq D$

Superset and subset: Exercise

- What is the relation between the sets in (16)?

(16) a. $B = \{x \mid \text{x is a Simpson}\}$

b. $C = \{y \mid \text{y is a fictional character on TV}\}$

- What is the relation between the sets in (17)?

(17) a. $A = \{a, y, \{2, d\}, \beta, \heartsuit, 3\}$

b. $F = \{y, \beta\}$

c. $G = \{y, 4, \beta\}$

- solutions:

(18) $B \subseteq C$ and $C \supseteq B$

(19) a. $F \subseteq A$ and $A \supseteq F$

b. $G \not\subseteq A$ and $A \not\supseteq G$

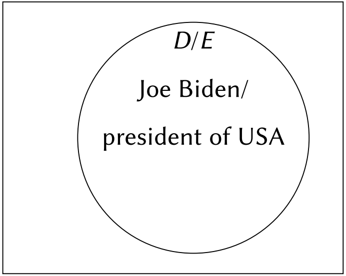
c. $F \subseteq G$ and $G \supseteq F$

Proper subset

- consider the following sets:

(20) a. $D = \{\text{Joe Biden}\}$

b. $E = \{x \mid \text{x is the current president of the USA}\}$

(21) 

$D = E$
because they have identical membership: the current president of the USA describes the same individual as Joe Biden, so . . .

$D \subseteq E$ and $E \subseteq D$

Proper subset

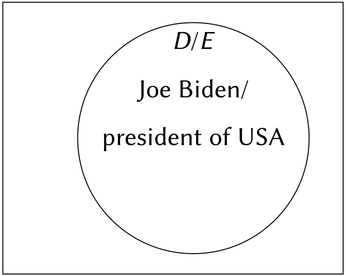
- why? let us go back to our definition:

(22) For any two sets X and Y, if every member of X is a member of Y, then:

a. X is a subset of Y, and

b. Y is a superset of X

(23)



$D \subseteq E$ and $E \subseteq D$
because **every member of D is also a member of E and every member of E is also a member of D**

hence: **every set is a subset of itself** (namely the subset that is identical to the set)

Proper subset

- if we want to exclude the case of a set being a subset of itself, we need to define a **proper subset**

(24) A set X is a proper subset of a set Y if:

  a. every member of X is also a member of Y, but

  b. not every member of Y is a member of X (i.e. Y has at least one member that is not in X)

- many of the set relations we have looked at so far qualify as proper subset relations
- we write $X \subset Y$ to indicate proper subsethood
- general subsethood is still written as $X \subseteq Y$
- the difference between proper subset $\subset$ and subset $\subseteq$ is the additional $=$ which indicates that only proper subsets exclude identity

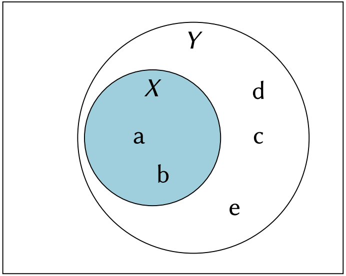
Proper subset

- an example:

(25) a. $X = \{a, b\}$

b. $Y = \{a, b, c, d, e\}$

(26)



X is a subset of Y because every member of X is also a member of Y ($X \subseteq Y$)

but X is also a proper subset of Y because not every member of Y is a member of X, so . . .

$X \subset Y$

Superset, subset, proper subset: Exercise

- What is the relation between the sets in (27)?

(27) a. $B = \{x \mid \text{x is a Simpson}\}$

b. $C = \{y \mid \text{y is a fictional character on TV}\}$

- What is the relation between the sets in (28)?

(28) a. $A = \{a, y, \{2, d\}, \beta, \heartsuit, 3\}$

b. $F = \{y, \beta\}$

c. $G = \{y, 4, \beta\}$

- What is the relation between the sets in (29)?

(29) a. $G = \{y, 4, \beta\}$

b. $H = \{y, 4, \beta\}$

- solutions:

(30) a. $B \subseteq C$ and $C \supseteq B$

b. $B \subset C$

(31) a. $F \subseteq A$ and $A \supseteq F$

b. $F \subset A$

c. $G \not\subseteq A$ and $A \not\supseteq G$

d. $F \subseteq G$ and $G \supseteq F$

e. $F \subset G$

(32) a. $G \subseteq H$ and $H \subseteq G$

b. $G \not\subset H$ and $H \not\subset G$