Formale Grundlagen (Logik)
Modul 04-006-1001

Predicate Logic IV

Leipzig University

February 1st, 2024

Fabian Heck

(Slides based on Partee, ter Meulen und Wall 1990
"Mathematical Methods in Linguistics")

Recap: Vocabulary of predicate logic

1. individual constants: $\{dan, norbert, m, j, \ldots\} \rightarrow$ *terms*
2. individual variables: $\{x, y, z, \ldots\} \rightarrow$ *terms*
3. predicate constants: $\{C, D, L, \ldots\}$
4. connectives: $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$
5. auxiliary symbols: (,), [,]
6. quantifiers: $\forall$ universal, $\exists$ existential

Recap: Syntax of predicate logic

(1) *Syntax of predicate logic*

a. If δ is an *n*-place predicate and $t_1, \ldots, t_n$ are terms, then $\delta(t_1, \ldots, t_n)$ is a well-formed formula.

b. If ϕ is a formula, then $(\neg\phi)$ is a well-formed formula.

c. If ϕ and ψ are formulas, then $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, and $(\phi \leftrightarrow \psi)$ are well-formed formulas.

d. If ϕ is a formula and x is an individual variable, then $(\forall x)\phi$ and $(\exists x)\phi$ are well-formed formulas.

e. Nothing else is a formula.

Recap: Semantics of predicate logic

(2) *Semantics of predicate logic*

a. If δ is an n-place predicate and $t_1, \ldots, t_n$ are terms, then $[\![\delta(t_1, \ldots, t_n)]\!]^M = 1$ iff $\langle [\![t_1]\!]^M, \ldots, [\![t_n]\!]^M \rangle \in [\![\delta]\!]^M$.

b. If ϕ is a formula, then $[\![(\neg\phi)]\!]^M = 1$ iff $[\![\phi]\!]^M = 0$.

c. If ϕ and ψ are formulas, then $[\![(\phi \wedge \psi)]\!]^M = 1$ iff both $[\![\phi]\!]^M = 1$ and $[\![\psi]\!]^M = 1$.

d. If ϕ and ψ are formulas, then $[\![(\phi \vee \psi)]\!]^M = 1$ iff at least one of $[\![\phi]\!]^M$, $[\![\psi]\!]^M = 1$.

e. If ϕ and ψ are formulas, then $[\![(\phi \rightarrow \psi)]\!]^M = 1$ iff either $[\![\phi]\!]^M = 0$ or $[\![\psi]\!]^M = 1$.

f. If ϕ and ψ are formulas, then $[\![(\phi \leftrightarrow \psi)]\!]^M = 1$ iff $[\![\phi]\!]^M = [\![\psi]\!]^M$.

Recap: Semantics of predicate logic with quantifiers

- the trick: let the open statement in the scope of the quantifier assume a truth-value temporarily
- specifically, we let the variable take on the values of all the individuals in the domain D one by one and see what truth value the statement would have for each of these individuals

(3) a. $[\![(\forall x)H(x)]\!]^M$ = TRUE iff $[\![x]\!]^M \in [\![H]\!]^M$ for all individuals in D; if for even one element in D, $[\![x]\!]^M \notin [\![H]\!]^M$, then $[\![(\forall x)H(x)]\!]^M = FALSE$

b. $[\![(\exists x)H(x)]\!]^M$ = TRUE iff $[\![x]\!]^M \in [\![H]\!]^M$ for at least one individual in D; if for no element in D, $[\![x]\!]^M \in [\![H]\!]^M$, then $[\![(\exists x)H(x)]\!]^M = FALSE$

Semantics of quantifiers: assignment functions

- the rules in (4-a,b) both make reference to $[\![x]\!]^M$, i.e. the denotation of a variable
- strictly speaking, such a denotation is not available in our models so far
- what the wordings "for all individuals in *D*"/"for at least one individual in *D*", respectively, suggest, is that we need a means to have a variable assume one or more particular values

(4) a. $[\![(\forall x)H(x)]\!]^M$ = TRUE iff $[\![x]\!]^M \in [\![H]\!]^M$ for all individuals in *D*; if for even one element in *D*, $[\![x]\!]^M \notin [\![H]\!]^M$, then $[\![(\forall x)H(x)]\!]^M = \mathit{FALSE}$

b. $[\![(\exists x)H(x)]\!]^M$ = TRUE iff $[\![x]\!]^M \in [\![H]\!]^M$ for at least one individual in *D*; if for no element in *D*, $[\![x]\!]^M \in [\![H]\!]^M$, then $[\![(\exists x)H(x)]\!]^M = \mathit{FALSE}$

Semantics of quantifiers: assignment functions

- to this end, we add an assignment function g ("Belegungsfunktion") to the model, the purpose of which is to assign a particular value to each variable

(5) $M = \langle D, I \rangle$, where:

a. $D = \{Mary, Jane, MMil\}$

b. $I =$

term	value	predicate	value
m	$Mary$	S	$\{Mary, Jane\}$
j	$Jane$	B	$\{MMil\}$
$mmil$	$MMil$	R	$\{\langle Mary, MMil \rangle\}$

c. $g(x) = Jane$, $g(y) = Mary$

(6) *Modified assignment*:
A modified assignment function $g' = g^{d/v}$ is a function that behaves exactly like g except that it assigns the individual $d \in D$ to the variable v.

Semantics of quantifiers: assignment functions

Semantics of predicate logic (part I):

(7) a. If α is a non-logical constant, then $[\![\alpha]\!]^{M,g} = I(\alpha)$

b. If α is a variable, then $[\![\alpha]\!]^{M,g} = g(\alpha)$

(8) If δ is an n-place predicate and $t_1, \ldots, t_n$ are terms, then $[\![\delta(t_1, \ldots, t_n)]\!]^{M,g} = 1$ iff $\langle [\![t_1]\!]^{M,g}, \ldots, [\![t_n]\!]^{M,g} \rangle \in [\![\delta]\!]^{M,g}$.

(9) For any formula ϕ, ψ:

a. $[\![(\neg\phi)]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 0$.

b. $[\![(\phi \wedge \psi)]\!]^{M,g} = 1$ iff both $[\![\phi]\!]^{M,g} = 1$ and $[\![\psi]\!]^{M,g} = 1$.

c. $[\![(\phi \vee \psi)]\!]^{M,g} = 1$ iff at least one of $[\![\phi]\!]^{M,g}$, $[\![\psi]\!]^{M,g} = 1$.

d. $[\![(\phi \rightarrow \psi)]\!]^{M,g} = 1$ iff either $[\![\phi]\!]^{M,g} = 0$ or $[\![\psi]\!]^{M,g} = 1$.

e. $[\![(\phi \leftrightarrow \psi)]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = [\![\psi]\!]^{M,g}$.

Semantics of quantifiers: assignment functions

Semantics of predicate logic (part II):

(10) a. If ϕ is a formula, and x is a variable, then $[\![(\exists x)\phi]\!]^{M,g} = 1$ iff there is at least one $d \in D$ such that $[\![\phi]\!]^{M,g^{d/x}} = 1$

b. If ϕ is a formula, and x is a variable, then $[\![(\forall x)\phi]\!]^{M,g} = 1$ iff for all $d \in D$, $[\![\phi]\!]^{M,g^{d/x}} = 1$

(11) For any formula ϕ, $[\![\phi]\!]^{M} = 1$ iff for all assignments g, $[\![\phi]\!]^{M,g} = 1$.

Semantics of quantifiers: assignment functions

- Illustration: the calculation of $[\![(12\text{-b})]\!]^{M,g}$

(12) a. Mary is reading a book.

b. $(\exists x)(book(x) \wedge read(m, x))$

(13) $M = \langle D, I \rangle$, where:

a. $D = \{Mary, Jane, MMil\}$

b. $I =$

term	value	predicate	value
m	*Mary*	*student*	$\{Mary, Jane\}$
j	*Jane*	*book*	$\{MMil\}$
mmil	*MMil*	*read*	$\{\langle Mary, MMil \rangle\}$

c. $g(x) = Jane$, $g(y) = Mary$

Semantics of quantifiers: assignment functions

(14) $[\![(\exists x)(book(x) \wedge read(m,x))]\!]^{M,g}$

(15) a. $[\![book]\!]^{M,g} = I(book) = \{MMil\}$, and $[\![x]\!]^{M,g} = g(x) = Jane$
(rules (7-a) and (7-b))

b. $[\![book(x)]\!]^{M,g} = 0$ because $Jane \notin [\![book]\!]^{M,g}$
(rule 8)

c. $[\![read]\!]^{M,g} = I(read) = \{\langle Mary, MMil \rangle\}$, and $[\![m]\!]^{M,g} = I(m) = Mary$
(2$\times$ rule (7-a))

d. $[\![read(m,x)]\!]^{M,g} = 0$, because $\langle Mary, Jane \rangle \notin [\![read]\!]^{M,g}$
(rule 8)

e. $[\![(book(x) \wedge read(m,x))]\!]^{M,g} = 0$
(rule (9-b), applied to (15-b) and (15-d))

Semantics of quantifiers: assignment functions

- to compute $[\![(\exists x)(book(x) \wedge read(m,x))]\!]^{M,g}$, we must check whether there is a modified assignment $g' = g^{d/x}$ such that $[\![(book(x) \wedge read(m,x))]\!]^{M,g^{d/x}} = 1$ for some $d \in D$ (rule (10-a))
- there is such an assignment: $g'(x) = MMil$:

(16) a. $[\![(book(x) \wedge read(m,x))]\!]^{M,g'} = 1$ because $g'(x) = MMil$, and $MMil \in [\![book]\!]^{M,g'}$, and $\langle Mary, MMil \rangle \in [\![read]\!]^{M,g'}$

b. $[\![(\exists x)(book(x) \wedge read(m,x))]\!]^{M,g} = 1$

- at this point, one may note that the computation would not have been different if we had started with an assignment function different from g in the first place
- in other words: the truth value of $(\exists x)(book(x) \wedge read(m,x))$ is independent from the assignment function it is evaluated against: the only variable it contains is bound
- therefore (rule (11)):

(17) $[\![(\exists x)(book(x) \wedge read(m,x))]\!]^{M} = 1$

Semantics of quantifiers: assignment functions

- Illustration: the calculation of $[\![(18\text{-b})]\!]^{M,g}$

(18) a. Every student is reading a book.

b. $(\forall y)(student(y) \rightarrow (\exists x)(book(x) \wedge read(y,x)))$

(19) $M = \langle D, I \rangle$, where:

a. $D = \{Mary, Jane, MMil\}$

b. $I =$

term	value	predicate	value
m	*Mary*	*student*	$\{Mary, Jane\}$
j	*Jane*	*book*	$\{MMil\}$
mmil	*MMil*	*read*	$\{\langle Mary, MMil \rangle\}$

c. $g(x) = Jane$, $g(y) = Mary$

Semantics of quantifiers: assignment functions

(20) $[\![(\forall y)(student(y) \rightarrow (\exists x)(book(x) \wedge read(y,x)))]\!]^{M,g}$

(21) a. $[\![book]\!]^{M,g} = I(book) = \{MMil\}$, and $[\![x]\!]^{M,g} = g(x) = Jane$
(rules (7-a) and (7-b))

b. $[\![book(x)]\!]^{M,g} = 0$ because $Jane \notin [\![book]\!]^{M,g}$
(rule 8)

c. $[\![read]\!]^{M,g} = I(read) = \{\langle Mary, MMil\rangle\}$, and $[\![y]\!]^{M,g} = g(y) = Mary$
(rules (7-a) and (7-b))

d. $[\![read(y,x)]\!]^{M,g} = 0$, because $\langle Mary, Jane\rangle \notin [\![read]\!]^{M,g}$
(rule 8)

e. $[\![(book(x) \wedge read(y,x))]\!]^{M,g} = 0$
(rule (9-b), applied to (21-b) and (21-d))

Semantics of quantifiers: assignment functions

- to compute $[\![(\exists x)(book(x) \wedge read(m,x))]\!]^{M,g}$, we must check whether there is a modified assignment $g' = g^{d/x}$ such that $[\![(book(x) \wedge read(m,x))]\!]^{M,g^{d/x}} = 1$ for some $d \in D$ (rule (10-a))
- again, there is such an assignment: $g'(x) = MMil$:

(22) a. $[\![(book(x) \wedge read(m,x))]\!]^{M,g'} = 1$ because $g'(x) = MMil$, and $MMil \in [\![book]\!]^{M,g'}$, and $\langle Mary, MMil \rangle \in [\![read]\!]^{M,g'}$

b. $[\![(\exists x)(book(x) \wedge read(m,x))]\!]^{M,g} = 1$

- we continue:

(23) a. $[\![student]\!]^{M,g} = I(student) = \{Mary, Jane\}$, and $[\![y]\!]^{M,g} = g(y) = Mary$ (rules (7-a) and (7-b))

b. $[\![student(y)]\!]^{M,g} = 1$ because $Mary \in [\![student]\!]^{M,g}$ (rule (8))

c. $[\![(student(y) \rightarrow (\exists x)(book(x) \wedge read(y,x)))]\!]^{M,g} = 1$ (rule (9-d), applied to (22-b) and (23-b))

Semantics of quantifiers: assignment functions

- $[\![(\forall y)(student(y) \rightarrow (\exists x)(book(x) \wedge read(m,x)))]\!]^{M,g}$: we check whether there is a $g' = g^{d/y}$ such that $[\![(student(y) \rightarrow (\exists x)(book(x) \wedge read(m,x)))]\!]^{M,g^{d/y}} = 1$ for all $d \in D$ (rule (10-b))
- let us try the assignment $g'' = g'^{Jane/y} = g^{MMil/x^{Jane/y}}$
- that is, let us see whether by further modifying our previous g', we can get $[\![(student(y) \rightarrow (\exists x)(book(x) \wedge read(m,x)))]\!]^{M,g''} = 1$, assuming that $g''(y) = Jane$:

(24) a. $[\![student(y)]\!]^{M,g''} = 1$ because $g''(y) = Jane$, and $Jane \in [\![student(y)]\!]^{M,g''}$ (rules (7-b) and (8))

b. $[\![(book(x) \wedge read(y,x))]\!]^{M,g''} = 0$ because $g''(x) = MMil$, $MMil \in [\![book]\!]^{M,g''}$, and $\langle Jane, MMil \rangle \notin [\![read]\!]^{M,g''}$ (rules (7-b) and 2× (8))

c. $[\![(\exists x)(book(x) \wedge read(y,x)))]\!]^{M,g''} = 0$ because there is no <u>other</u> function $g''' = g''^{d/x}$ for some $d \in D$ such that $[\![(book(x) \wedge read(y,x))]\!]^{M,g'''} = 1$

d. $[\![(student(y) \rightarrow (\exists x)(book(x) \wedge read(m,x)))]\!]^{M,g''} = 0$ because of rule (9-d) applied to (24-a) and (24-c)

e. $[\![(\forall y)(student(y) \rightarrow (\exists x)(book(x) \wedge read(m,x)))]\!]^{M,g} = 0$ (rule (10-b) and (24-d))