Formale Grundlagen (Logik)
Modul 04-006-1001

Predicate Logic III

Leipzig University

January 25^{th}, 2024

Fabian Heck
(Slides by Imke Driemel & Sandhya Sundaresan,
based on Partee, ter Meulen und Wall 1990
“Mathematical Methods in Linguistics”)

Recap: Vocabulary of predicate logic

- predicate logic has the following components:
  1. individual constants: $\{dan, norbert, m, j, \dots\} \rightarrow$ *terms*
  2. individual variables: $\{x, y, z, \dots\} \rightarrow$ *terms*
  3. predicate constants: $\{C, D, L, \dots\}$
  4. connectives: $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$
  5. auxiliary symbols: (,), [,]
  6. quantifiers: $\forall$ universal, $\exists$ existential
- these various pieces are combined to make up expressions (atomic or complex) and (open) statements

Recap: Syntax of predicate logic

- the first rule creates atomic formulas (formulas without connectives or quantifiers)
- the second and third rules create complex (open) statements

(1) *Syntax of predicate logic*

a. If δ is an n-place predicate and $t_1, \ldots, t_n$ are terms, then $\delta(t_1, \ldots, t_n)$ is a well-formed formula.

b. If ϕ is a formula, then $(\neg\phi)$ is a well-formed formula.

c. If ϕ and ψ are formulas, then $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, and $(\phi \leftrightarrow \psi)$ are well-formed formulas.

d. If ϕ is a formula and x is an individual variable, then $(\forall x)\phi$ and $(\exists x)\phi$ are well-formed formulas.

e. Nothing else is a formula.

Recap: Semantics of predicate logic

- as with statement logic, a statement in predicate logic can have one of two binary values: True or False (relative to a model M)
- except for the first rule, the semantic rules are identical to those of statement logic

(2) *Semantics of predicate logic*

a. If δ is an n-place predicate and $t_1, \ldots, t_n$ are terms, then $[\![\delta(t_1, \ldots, t_n)]\!]^M = 1$ iff $\langle [\![t_1]\!]^M, \ldots, [\![t_n]\!]^M \rangle \in [\![\delta]\!]^M$.

b. If ϕ is a formula, then $[\![(\neg\phi)]\!]^M = 1$ iff $[\![\phi]\!]^M = 0$.

c. If ϕ and ψ are formulas, then $[\![(\phi \wedge \psi)]\!]^M = 1$ iff both $[\![\phi]\!]^M = 1$ and $[\![\psi]\!]^M = 1$.

d. If ϕ and ψ are formulas, then $[\![(\phi \vee \psi)]\!]^M = 1$ iff at least one of $[\![\phi]\!]^M$, $[\![\psi]\!]^M = 1$.

e. If ϕ and ψ are formulas, then $[\![(\phi \rightarrow \psi)]\!]^M = 1$ iff either $[\![\phi]\!]^M = 0$ or $[\![\psi]\!]^M = 1$.

f. If ϕ and ψ are formulas, then $[\![(\phi \leftrightarrow \psi)]\!]^M = 1$ iff $[\![\phi]\!]^M = [\![\psi]\!]^M$.

- there are certain statements whose truth value is fixed, regardless of the model

(3) a. $P(s) \vee \neg P(s)$ is always True (logical tautology)

b. $P(s) \wedge \neg P(s)$ is always False (logical contradiction)

Recap: Semantics of predicate logic

A model M is a pair $\langle D, I \rangle$, where D is the domain, a set of individuals, and I is an interpretation function: an assignment of semantic values to every basic expression in the language

(4) $M = \langle D, I \rangle$, where:

a. $D = \{Dee, Nat, Jean, Mo\}$

b. I determines the following mapping between terms and predicate terms on the one hand, and objects in D/ sets of (tuples of) objects on the other hand:

term	value	predicate	value
d	Dee	H	$\{Nat, Mo\}$
n	Nat	C	$\{Nat, Mo, Dee\}$
j	Jean	R	$\{\langle Mo, Dee \rangle, \langle Nat, Dee \rangle\}$
m	Mo	L	$\{\langle Nat, Jean \rangle, \langle Dee, Jean \rangle, \langle Mo, Jean \rangle\}$

(5) A model M consist of a set D and a function I which assigns:

a. to each individual constant, a member of D

b. to each one-place predicate, a subset of D

c. to each two-place predicate, a subset of $D \times D$

d. to each n-place predicate, a subset of $D_1 \times D_2 \times \cdots \times D_n$

Recap: Semantics of predicate logic

(6) Semantic rules of predicate logic

a. If δ is an n-place predicate and $t_1, \ldots, t_n$ are terms, then $[\![\delta(t_1, \ldots, t_n)]\!]^M = 1$ iff $\langle [\![t_1]\!]^M, \ldots, [\![t_n]\!]^M \rangle \in [\![\delta]\!]^M$.

b. If ϕ is a formula, then $[\![\neg\phi]\!]^M = 1$ iff $[\![\phi]\!]^M = 0$.

c. If ϕ and ψ are formulas, then $[\![\phi \wedge \psi]\!]^M = 1$ iff both $[\![\phi]\!]^M = 1$ and $[\![\psi]\!]^M = 1$.

d. If ϕ and ψ are formulas, then $[\![\phi \vee \psi]\!]^M = 1$ iff at least one of $[\![\phi]\!]^M$, $[\![\psi]\!]^M = 1$.

e. If ϕ and ψ are formulas, then $[\![\phi \rightarrow \psi]\!]^M = 1$ iff either $[\![\phi]\!]^M = 0$ or $[\![\psi]\!]^M = 1$.

f. If ϕ and ψ are formulas, then $[\![\phi \leftrightarrow \psi]\!]^M = 1$ iff $[\![\phi]\!]^M = [\![\psi]\!]^M$.

With respect to Model M, the truth values of the (well-formed) formulas below are as follows:

(7) a. $[\![C(j)]\!]^M$ **false**

b. $[\![R(n, d)]\!]^M$ **true**

c. $[\![C(j \wedge d)]\!]^M$ **not well-formed**

d. $[\![L(n, j) \vee C(j)]\!]^M$ **true**

e. $[\![(C(d) \wedge L(n, j)) \rightarrow H(n)]\!]^M$ **true**

term	value	pred	value
d	Dee	H	$\{Nat, Mo\}$
n	Nat	C	$\{Nat, Mo, Dee\}$
j	Jean	R	$\{\langle Mo, Dee\rangle, \langle Nat, Dee\rangle\}$
m	Mo	L	$\{\langle Nat, Jean\rangle, \langle Dee, Jean\rangle, \langle Mo, Jean\rangle\}$

Table: Model M

Quantifier scope and quantifier binding

- we will now introduce some terminology, recall the syntax of quantifiers:

(8) *Syntax of predicate logic*

d. If ϕ is a formula and x is an individual variable, then $(\forall x)\phi$ and $(\exists x)\phi$ are well-formed formulas.

- ϕ is called the **scope** of the quantifier (we can also say: ϕ lies in the scope of the quantifier)
- if there are no brackets, then the minimal formula immediately following the quantifier lies in its scope, brackets may be used to extend the scope of a quantifier

(9) a. $((\exists x)R(x, y) \wedge P(x))$

scope of $(\exists x)$

b. $(\exists x)(R(x, y) \wedge P(x))$

scope of $(\exists x)$

Quantifier scope and quantifier binding

- the scope of one quantifier may be contained in the scope of another quantifier

(10) $(\exists x)(\forall y)(R(x,y) \wedge P(x,x))$

scope of $(\exists x)$

scope of $(\forall y)$

- determine the scope for each of the quantifiers in the following example!

(11) $(\exists x)[Q(x) \wedge (\forall y)(P(y) \rightarrow (\exists z)S(x,y,z))]$

Quantifier scope and quantifier binding

- the scope of one quantifier may be contained in the scope of another quantifier

(10) $(\exists x)(\forall y)(R(x,y) \wedge P(x,x))$

scope of $(\exists x)$

scope of $(\forall y)$

- determine the scope for each of the quantifiers in the following example!

(11) $(\exists x)[Q(x) \wedge (\forall y)(P(y) \rightarrow (\exists z)S(x,y,z))]$

scope of $(\exists x)$

scope of $(\forall y)$

scope of $(\exists z)$

Quantifier scope and quantifier binding

(12) *Quantifier binding:*
An occurrence of a variable x is bound iff it occurs in the minimal scope of $(\exists x)$ or $(\forall x)$ (i.e., there is no closer binder available). A variable is free iff it is not bound.

- the existential quantifier $\exists x$ in (13-a) binds both instances of x
- a quantifier may also fail to bind any variable, for instance if there is no appropriate variable at all in its scope (13-b) (vacuous quantification)

(13) a. $(\exists x)(R(x, y) \wedge P(x))$
scope of $(\exists x)$

b. $(\exists x)(R(m, j) \wedge P(k))$
scope of $(\exists x)$

Quantifier scope and quantifier binding

- all open statements contain at least one free variable
- a statement is a formula that does not contain a free variable
- important: a variable is always bound by the closest quantifier indexed for it
- in the example below, the variable x in the open statement $M(x)$ is only bound by the existential quantifier $(\exists x)$ (although $M(x)$ is in the scope of both $(\exists x)$ and $(\forall x)$) since $(\exists x)$ is closer to $M(x)$ than $(\forall x)$

(14) $(\forall x)(R(x) \rightarrow (\exists x)M(x))$

scope of $(\exists x)$

scope of $(\forall x)$

- a perhaps less confusing (but logically equivalent) way of writing up the formula would be this:

(15) $(\forall x)(R(x) \rightarrow (\exists y)M(y))$

scope of $(\exists y)$

scope of $(\forall x)$

Semantics of predicate logic with quantifiers

- figuring out the truth-values of statements that contain quantifiers and variables is a little more complicated; the trick is to let the open statement in the scope of the quantifier assume a truth-value temporarily
- specifically, we let the variable take on the values of all the individuals in the domain D one by one and see what truth value the statement would have for each of these individuals

(16) a. $[\![(\forall x)H(x)]\!]^M$ = TRUE iff $[\![x]\!]^M \in [\![H]\!]^M$ for all individuals in D; if for even one element in D, $[\![x]\!]^M \notin [\![H]\!]^M$, then $[\![(\forall x)H(x)]\!]^M = FALSE$

b. $[\![(\exists x)H(x)]\!]^M$ = TRUE iff $[\![x]\!]^M \in [\![H]\!]^M$ for at least one individual in D; if for no element in D, $[\![x]\!]^M \in [\![H]\!]^M$, then $[\![(\exists x)H(x)]\!]^M = FALSE$

Exercise: semantics of predicate logic with quantifiers

(17) $M = \langle D, I \rangle$, where:

a. $D = \{Dee, Nat, Jean, Mo\}$

b. $I =$

term	value	predicate	value
d	Dee	H	$\{Nat, Mo\}$
n	Nat	C	$\{Nat, Mo, Dee\}$
j	Jean	R	$\{\langle Mo, Dee \rangle, \langle Nat, Dee \rangle\}$
m	Mo	L	$\{\langle Nat, Jean \rangle, \langle Dee, Jean \rangle, \langle Mo, Jean \rangle\}$

(18)

a. $[\![(\exists x)H(x)]\!]^M =$

b. $[\![(\exists x)L(x, j)]\!]^M =$

c. $[\![(\forall x)L(x, j)]\!]^M =$

d. $[\![(\exists y)L(d, y)]\!]^M =$

e. $[\![((\exists y)L(d, y) \wedge H(d))]\!]^M =$

(19)

a. $[\![((\exists y)H(m) \wedge L(d, y))]\!]^M =$

b. $[\![(\exists y)(H(d) \wedge L(d, y))]\!]^M =$

c. $[\![(\forall x)(C(x) \rightarrow L(x, j))]\!]^M =$

d. $[\![(\exists y)(H(z) \wedge C(z))]\!]^M =$

e. $[\![(\exists x)R(m, d)]\!]^M =$

Exercise: semantics of predicate logic with quantifiers

(18)
a. $[\![(\exists x)H(x)]\!]^M = \mathit{TRUE}$
b. $[\![(\exists x)L(x,j)]\!]^M = \mathit{TRUE}$
c. $[\![(\forall x)L(x,j)]\!]^M = \mathit{FALSE}$
d. $[\![(\exists y)L(d,y)]\!]^M = \mathit{TRUE}$
e. $[\![((\exists y)L(d,y) \wedge H(d))]\!]^M = \mathit{FALSE}$

(19)
a. $[\![((\exists y)H(m) \wedge L(d,y))]\!]^M = ?$
b. $[\![(\exists y)(H(d) \wedge L(d,y))]\!]^M = \mathit{FALSE}$
c. $[\![(\forall x)(C(x) \rightarrow L(x,j))]\!]^M = \mathit{TRUE}$
d. $[\![(\exists y)(H(z) \wedge C(z))]\!]^M = ?$
e. $[\![(\exists x)R(m,d)]\!]^M = \mathit{TRUE}$

- the truth values of (19-a) and (19-d) cannot be determined because they contain open statements
- in (19-a), the expression $L(d,y)$ is not in the scope of $(\exists y)$ (only $H(m)$ is), and therefore the variable y cannot be bound
- in (19-d), the expression $(H(z) \wedge C(z))$ is in the scope of $\exists$, but this quantifier is indexed with another variable (namely y)

Semantics of predicate logic with quantifiers

- in expressions with more than one quantifiers, the order of quantifiers is important: we start with the outermost quantifier and then work our way inwards

(20) $M' = \langle D, I \rangle$, where:

a. $D = \{Dee, Nat, Jean, Mo\}$

b. $I =$

term	value	predicate	value
d	Dee	H	$\{Nat, Mo\}$
n	Nat	C	$\{Nat, Mo, Dee\}$
j	Jean	R	$\{\langle Mo, Dee\rangle, \langle Nat, Dee\rangle\}$
m	Mo	L	$\{\langle Nat, Jean\rangle, \langle Dee, Jean\rangle, \langle Mo, Jean\rangle, \langle Jean, Nat\rangle\}$

(21) $[\![(\forall x)(\exists y)L(x,y)]\!]^{M'} = TRUE$ (because everyone loves someone)
the entire expression is TRUE iff for every possible value of x in D, $[\![(\exists y)L(x,y)]\!]^{M'}$ is TRUE; otherwise it is FALSE

(22) $[\![(\exists y)(\forall x)L(x,y)]\!]^{M'} = FALSE$ (because Jean doesn't love himself)
the entire expression is TRUE iff for at least one value of y in D, $[\![(\forall x)L(x,y)]\!]^{M'}$ is TRUE; otherwise it is FALSE

Exercise: semantics of predicate logic with quantifiers

(23) $M'' = \langle D, I \rangle$, where:

a. $D = \{Dee, Nat, Jean, Mo\}$

b. $I =$

term	value	predicate	value
d	Dee	H	$\{Nat, Mo\}$
n	Nat	C	$\{Nat, Mo, Dee\}$
j	Jean	R	$\{\langle Mo, Dee\rangle, \langle Nat, Dee\rangle, \langle Jean, Dee\rangle, \langle Dee, Dee\rangle\}$
m	Mo	L	$\{\langle Nat, Jean\rangle, \langle Dee, Jean\rangle, \langle Mo, Jean\rangle\}$

(24) a. $[\![(\forall x)(\exists y)L(x, y)]\!]^{M''} =$

b. $[\![(\forall x)(\exists y)R(x, y)]\!]^{M''} =$

c. $[\![(\exists y)(\forall x)R(x, y)]\!]^{M''} =$

Exercise: semantics of predicate logic with quantifiers

(23) $M'' = \langle D, I \rangle$, where:

a. $D = \{Dee, Nat, Jean, Mo\}$

b. $I =$

term	value	predicate	value
d	Dee	H	$\{Nat, Mo\}$
n	Nat	C	$\{Nat, Mo, Dee\}$
j	Jean	R	$\{\langle Mo, Dee \rangle, \langle Nat, Dee \rangle, \langle Jean, Dee \rangle, \langle Dee, Dee \rangle\}$
m	Mo	L	$\{\langle Nat, Jean \rangle, \langle Dee, Jean \rangle, \langle Mo, Jean \rangle\}$

(24) a. $[\![(\forall x)(\exists y)L(x, y)]\!]^{M''} = FALSE$

b. $[\![(\forall x)(\exists y)R(x, y)]\!]^{M''} = TRUE$

c. $[\![(\exists y)(\forall x)R(x, y)]\!]^{M''} = TRUE$

Proofs in predicate logic

- our rules of inference need some additions to handle quantifiers
- one proof is called Existential Generalization: If we know one true instantiation of a formula, we can generalize to an existential statement

(25) Maria is Anwältin.
$\therefore$ Es gibt mindestens eine Anwältin.

(26)
$$\frac{A(maria)}{\therefore \quad (\exists x)A(x)}$$

(27)
$$\frac{\varphi(c)}{\therefore \quad (\exists x)\varphi(x)}$$

Proofs in predicate logic

- our rules of inference need some additions to handle quantifiers
- another proof is called **Universal Instantiation**: If we know that every value for a variable in a formula is true, we can infer to some particular instantiation

(28)
Alles ist vergänglich.
∴ Hans ist vergänglich.

(29)
$$\frac{(\forall x)V(x)}{\therefore \quad V(hans)}$$

(30)
$$\frac{(\forall x)\varphi(x)}{\therefore \quad \varphi(c)}$$

Proofs in predicate logic

(31) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(32) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(33) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(34) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(35) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(36) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(37) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

(38) **Exist. Gen.**

$$\begin{array}{ll} & \varphi(c) \\ \hline \therefore & (\exists x)\varphi(x) \end{array}$$

(39) **Uni. Inst.**

$$\begin{array}{ll} & (\forall x)\varphi(x) \\ \hline \therefore & \varphi(c) \end{array}$$

Alle Menschen sind sterblich.
Sokrates ist ein Mensch.

∴ Sokrates ist sterblich.

(40) *proof*:

1. $(\forall x)(M(x) \rightarrow S(x))$
2. $M(s)$
3. $(M(s) \rightarrow S(s))$ 1 UI
4. $(S(s))$ 2,3 MP

Proofs in predicate logic

(41) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(42) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(43) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(44) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(45) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(46) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(47) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

(48) **Exist. Gen.**

$$\begin{array}{ll} & \varphi(c) \\ \hline \therefore & (\exists x)\varphi(x) \end{array}$$

(49) **Uni. Inst.**

$$\begin{array}{ll} & (\forall x)\varphi(x) \\ \hline \therefore & \varphi(c) \end{array}$$

$$\begin{array}{ll} & \text{Alle sind verwundbar.} \\ \hline \therefore & \text{Einer ist verwundbar.} \end{array}$$

(50) *proof*:

1. $(\forall x)V(x)$
2. $V(m)$ — 1 UI
3. $(\exists x)V(x)$ — 2 EG