Formale Grundlagen (Logik)
Modul 04-006-1001

Predicate Logic I

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(Slides by Imke Driemel & Sandhya Sundaresan,
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“Mathematical Methods in Linguistics”)

Recap: Direct conditional proofs

- if the conclusion of a proof contains a conditional as the main connective, we can use a method of argumentation called **conditional proof**
  - suppose a proof has premises: $p_1, p_2, \ldots, p_n$ and $(q \rightarrow r)$ as the conclusion
  - in the conditional proof, we add the antecedent q of the conclusion as an additional **auxiliary premise**
  - we then derive r from the premises $p_1, p_2, \ldots, p_n$ and the auxiliary premise q
- the validity of the conditional proof is based on the following logical equivalence:

(1) $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow (q \rightarrow r) \Leftrightarrow (p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge q) \rightarrow r$

Recap: Direct conditional proofs

- a conditional proof is called conditional because it relies on an additional premise
- the conclusion is true under the condition that the auxiliary assumption is true

  - we indicate with a vertical bar every line of the proof which is based on the auxiliary premise
  - this is to remind ourselves that we are working with an additional special assumption
  - it is very important to always cancel that auxiliary premise by the rule of Conditional Proof before ending the entire proof (means going back to original position)
  - under the assumption that p is true, the conditional $(p \rightarrow q)$ holds
  - $(p \rightarrow q)$ does not follow directly from premises 1. and 2.

Given the premises 1.-2. we can prove $(p \rightarrow q)$!

(2) *conditional proof*:

1.	$(p \rightarrow (q \vee r))$	
2.	$(\neg r)$	
3.	$\mid\ p$	Aux
4.	$\mid\ (q \vee r)$	1,3 MP
5.	$\mid\ (r \vee q)$	4 Comm
6.	$\mid\ q$	2,5 DS
7.	$(p \rightarrow q)$	3-6 CP

Recap: Indirect conditional proofs

- indirect proofs aim at contradictions
- this form of argumentation uses the logic of **reductio ad absurdum**, which we have seen earlier
  - we still start with premise p, but we introduce the **negation of the conclusion**, i.e. $(\neg q)$ **as an auxiliary premise**
  - then we try to derive a contradiction
  - if we derive a contradiction, then we have indirectly shown that q does follow from p by showing that $(\neg q)$ is not compatible with p
  - if we don't derive a contradiction, then we have indirectly shown that the proof is not valid after all and that q does not follow from p
- an indirect proof is a type of conditional proof since it uses an auxiliary premise
- unlike in the conditional proofs seen earlier, the auxiliary premise here is not a part of the conclusion: rather, it is the negation of the conclusion
- also: the conclusion itself doesn't need to be in a conditional form, it could even be an atomic statement

Exercise: Indirect conditional proofs

(3) **Modus Ponens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & p \\ \hline \therefore & q \end{array}$$

(4) **Modus Tollens**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (\neg q) \\ \hline \therefore & (\neg p) \end{array}$$

(5) **Hyp. Syll.**

$$\begin{array}{ll} & (p \rightarrow q) \\ & (q \rightarrow r) \\ \hline \therefore & (p \rightarrow r) \end{array}$$

(6) **Dis. Syll.**

$$\begin{array}{ll} & (p \vee q) \\ & (\neg p) \\ \hline \therefore & q \end{array}$$

(7) **Simplification**

$$\begin{array}{ll} & (p \wedge q) \\ \hline \therefore & p \end{array}$$

(8) **Conjunction**

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & (p \wedge q) \end{array}$$

(9) **Addition**

$$\begin{array}{ll} & p \\ \hline \therefore & (p \vee q) \end{array}$$

Given the premises 1.-3. we can prove m!

(10) *indirect proof*:

1.	$(\neg m) \rightarrow (n \wedge o)$	
2.	$(n \rightarrow p)$	
3.	$(o \rightarrow (\neg p))$	
4.	$\mid (\neg m)$	Aux
5.	$\mid (n \wedge o)$	1,4 MP
6.	$\mid n$	5 Simpl
7.	$\mid o$	5 Simpl
8.	$\mid p$	2,6 MP
9.	$\mid (\neg p)$	3,7 MP
10.	$\mid (p \wedge (\neg p))$	8,9 Conj
11.	m	4-10 IP

New topic

Predicate Logic

Statement logic vs. predicate logic

- so far, we have learned the language of statement logic
- we will now look at a slightly more complex logical language, namely **predicate logic**
- in statement logic, the atoms (or primitives) are entire statements
- in predicate logic, on the other hand, the atoms come in two parts: **predicates** and **terms**
- hence, predicate logic is more fine-grained than statement logic: it looks inside a statement to analyze its logical structure and meaning
- with predicate logic, we will be able to construct proofs like:

(11)

	All linguists are logicians.
	Mary is a linguist.
$\therefore$	Mary is a logician.

Predicate logic

- a term is either a **constant** or a **variable**; these are expressions about which statements are made
  - a constant refers to a fixed entity, e.g. an individual like *John*
  - a variable does not refer to a fixed entity
- **predicates** express properties of terms, they can be understood as functions

(12) Max is small. $\underbrace{SMALL}_{predicate}(\underbrace{max}_{constant})$

- we represent predicates with capital letters: $S, P, Q, R, \ldots$, and terms with lower-case letters, where variables are often $x, y, z, \ldots$ and constants are either written out $max, peter, \ldots$ or abbreviated $m, p, \ldots$

(13) Max is small $= S(max) = S(m)$

Predicate logic

- when predicates combine with constants, they form statements
- predicates have different arities, also called the predicates' valencies
- a predicate may be ...
  - *one-place*: Susan is human $= H(susan)$
  - *two-place*: John loves Peter $= L(john, peter)$
  - *three-place*: Lee introduces Jane to Mo. $= I(lee, mo, jane)$
- ... or more, depending on the number of terms it combines with
- in general, an n-place predicate is one that combines with n terms to form a statement
- if the arity of a predicate doesn't match the number of terms it combines with, the result is ill-formed
- the linear order in which terms show up as arguments of a predicate is important: $I(lee, mo, jane) \neq I(lee, jane, mo)$

Exercise

- What's the predicate and how many terms does it combine with?

(14) a. Hans schnarcht.

b. Anna hilft Hans.

c. Maria ist die Schwester von Fritz.

d. Hans ist Anna ähnlicher als Fritz Maria.

e. Die Sonne scheint.

f. Berlin liegt zwischen Warschau und Paris.

g. Der Rhein ist ein Fluss.

Exercise

- What's the predicate and how many terms does it combine with?

(14)

(14)	a.	Hans schnarcht.	*one-place: schnarcht*
	b.	Anna hilft Hans.	*two-place: hilft*
	c.	Maria ist die Schwester von Fritz.	*two-place: ist Schwester von*
	d.	Hans ist Anna ähnlicher als Fritz Maria.	*four-place: ist ähnlicher als*
	e.	Die Sonne scheint.	*one-place: scheint*
	f.	Berlin liegt zwischen Warschau und Paris.	*three-place: liegt zwischen*
	g.	Der Rhein ist ein Fluss.	*one-place: ist ein Fluss*

Predicate logic

- like in statement logic, we can form complex expressions by using connectives

(15) Jumbo ist größer als Max oder Max ist größer als Jumbo.
$= G(jumbo, max) \vee G(max, jumbo)$

- what would be the predicate logic equivalents for the following statements?

(16) a. Jumbo ist größer oder kleiner als Max.
b. Fritz ist nicht reich.
c. Hans und Peter sind Studenten.
d. Hans und Peter sind Freunde.
e. Felix ist ein gelber Papagei.
f. Wenn Max Hans sieht, dann lacht Hans immer.
g. Anna und Fritz bewundern einander.
h. Fritz bewundert sich nicht selbst.

Predicate logic

- like in statement logic, we can form complex expressions by using connectives

(15) Jumbo ist größer als Max oder Max ist größer als Jumbo.
$= G(jumbo, max) \vee G(max, jumbo)$

- what would be the predicate logic equivalents for the following statements?

(16)
a. Jumbo ist größer oder kleiner als Max. $= G(j, m) \vee K(j, m)$
b. Fritz ist nicht reich. $= \neg R(f)$
c. Hans und Peter sind Studenten. $= S(h) \wedge S(p)$
d. Hans und Peter sind Freunde. $= F(h, p) \wedge F(p, h))$
e. Felix ist ein gelber Papagei. $= G(f) \wedge P(f)$
f. Wenn Max Hans sieht, dann lacht Hans immer. $= S(m, h) \rightarrow L(h)$
g. Anna und Fritz bewundern einander. $= B(a, f) \wedge B(f, a)$
h. Fritz bewundert sich nicht selbst. $= \neg B(f, f)$

Quantifiers

- so far we only looked at constants, but we said terms can also be variables
- when a predicate combines with a variable (or variables, depending on its arity), the result is not a statement but an expression called an open statement or propositional function
- $SLEEP(x)$, where x is a variable, does not correspond to a statement in English
- we can convert an open statement into a statement by adding something extra to the beginning of the expression: a logical primitive called a **quantifier** (plus the variable of the open statement)
- we consider two types of quantifiers:
    1. the **universal quantifier**, denoted by the symbol $\forall$, and corresponding to English expressions like *all, each*, and *every*
    2. the **existential quantifier**, denoted by the symbol $\exists$, and corresponding to English expressions like *some, at least one, ...*
- $SLEEP(x)$ can be converted into a statement by adding a quantifier that **binds** the variable as follows:

(17) a. $(\forall x)S(x)$ = Everyone sleeps. (read: "For all x, it is the case that $S(x)$")

b. $(\exists x)S(x)$ = Someone sleeps. (read: "There is some x, such that $S(x)$")

Quantifiers

- let us look at the universal quantifier in more detail

(18) $(\forall x)S(x) =$ Everyone sleeps.

- (18) means that for all possible values for x the predicate S holds of x
- if there is even one individual who is not sleeping, then the statement would be false
- and now the existential quantifier:

(19) $(\exists x)S(x) =$ Someone sleeps.

- (19) means that there is at least one value for x such that the predicate S holds of x
- if there is no individual that sleeps, then the statement is false

Quantifiers

- we can extend the same reasoning to more complex predicates, like two-place, three-place and, in theory, any *n*-ary predicate

(20) a. $(\forall y)H(mary, y) =$ Mary hates everyone.

b. $(\exists y)H(mary, y) =$ Mary hates someone.

- the universal statement expresses that for all (possible values of) y the relation H holds with *mary*
- the existential statement expresses that there is at least one (possible value of) y with which the relation H with *mary* holds
- it is possible to have more than one variable (each of which will then be bound by some quantifier)

(21) $(\forall x)(\forall y)H(x, y) =$ Everyone hates everyone.

- this universal statement expresses that for every (possible value of) x and for every (possible value of) y, x stands in the relation S to y

Quantifiers

- let us consider an example involving the same predicate and two variables, but two different quantifiers

(22) $(\forall x)(\exists y)H(x, y) =$ Everyone hates someone.

- the statement expresses that for every x, there is at least one (i.e. some) y such that the relation H holds between x and y
- if there is one x who does not hate anyone (i.e., there is no y for this x such that x hates y), the statement is false
- what do the following expressions mean? ($L = LIKE$, $G = GIVE$)

(23) a. $(\exists x)(\exists y)L(x, y)$

b. $(\forall x)(\forall y)(\forall z)G(x, y, z)$

Quantifiers

- let us consider an example involving the same predicate and two variables, but two different quantifiers

(22) $(\forall x)(\exists y)H(x, y) =$ Everyone hates someone.

- the statement expresses that for every x, there is at least one (i.e. some) y such that the relation H holds between x and y
- if there is one x who does not hate anyone (i.e., there is no y for this x such that x hates y), the statement is false
- what do the following expressions mean? ($L = LIKE$, $G = GIVE$)

(23) a. $(\exists x)(\exists y)L(x, y) =$ Someone likes someone.

b. $(\forall x)(\forall y)(\forall z)G(x, y, z) =$ Everyone gives everything to everyone.

Quantifiers

- the choice of the variable letter is arbitrary, it is just a notation and does not have any deeper meaning
- we can replace variable letters without any effect on meaning (as long as the replacement is uniform); the following two formulas are **alphabetical variants**

(24) $(\forall x)(\exists y)H(x, y) \Leftrightarrow (\forall y)(\exists x)H(y, x)$

- why is there no equivalence relation for the following two formulas?

(25) $(\forall x)(\exists y)H(x, y) \not\Leftrightarrow (\forall y)(\exists x)H(x, y)$

- they are not alphabetical variants because the quantifiers are associated with different positions in the predicate H

(26) Everyone hates at least one person. $\neq$ Everyone is hated by at least one person.

Quantifiers

- there is also no equivalence of the following two formulas

(27) $(\forall x)(\exists y)H(x,y) \not\Leftrightarrow (\exists y)(\forall x)H(x,y)$

- in this case, the order of the quantifiers was changed, which also changes the meaning (if the quantifiers are of different types: $\exists/\forall$)

(28) Everyone hates at least one person. $\neq$ At least one person is hated by everyone.

- in the first case, the hated person may be a different one for every hater (although it may also be the same), in the second, it must be one and the same hated person for all haters

Quantifiers

- and there is also no equivalence of the following two formulas

(29) $(\forall x)(\exists y)H(x,y) \not\Leftrightarrow (\forall x)(\exists y)H(y,x)$

- here, again, the quantifiers are associated with different positions in the predicate H ($\forall$ binds the subject vs. $\forall$ binds the object)

(30) Everyone hates at least one person. $\neq$ Everyone is hated by at least one person.

- finally, the following two formulas are not equivalent either

(31) $(\forall x)(\exists y)H(x,y) \not\Leftrightarrow (\exists y)(\forall x)H(y,x)$

(32) Everyone hates at least one person. $\neq$ At least one person hates everyone.

Quantifiers

- summary so far:

(33) a. $(\forall x)P(x)$: "For every x, P of x is true."

b. $(\exists x)P(x)$: "For at least one x, P of x is true."

- we also have a way of expressing *nobody, no one, nothing, . . .* within predicate logic

(34) $\neg(\exists x)P(x)$:
"It is not the case that for at least one x, P of x is true."

- here is an example:

(35) $\neg(\exists x)L(mary, x) =$ Mary likes nothing/no-one.

Quantifiers

- so far we have looked at quantifiers with the natural language equivalents *everyone, everything, someone, something, nothing, . . .*
- but what about quantifier expressions like *every woman, some child, no child, at least one apple*, etc.
- to get these, we have to add another predicate, the so called **restriction** (here the predicate W), to the quantifier

(36) a. Every woman ran.
$= (\forall x)(W(x) \rightarrow R(x)) =$ For every x, if x is female, then x ran.

b. Some woman ran.
$= (\exists x)(W(x) \wedge R(x)) =$ There is at least one x, such that x is female and x ran.

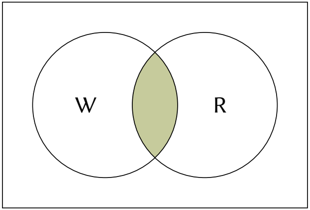
Quantifiers

- one-place predicates can also be seen as sets
- quantifiers then express relations between sets
- existential quantifiers say something about the intersection of two sets

(37) Some woman ran.
$= (\exists x)(W(x) \wedge R(x))$ = There is at least one x, such that x is female and x ran.

- in terms of set theory, this means that the intersection of the set of women and the set of individuals that run is not empty

(38)



(39) $W \cap R \neq \varnothing$

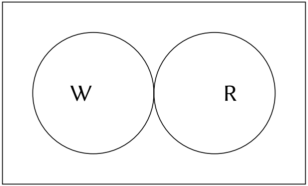
Quantifiers

- one-place predicates can also be seen as sets
- quantifiers express relations between sets
- the negated existential quantifier also says something about the intersection of two sets

(40) No woman ran.
$= \neg(\exists x)(W(x) \wedge R(x))$
$=$ It is not the case that there is at least one x, such that x is female and x ran.

- in terms of set theory, this means that the intersection of the set of women and the set of individuals that run is empty

(41)



(42) $W \cap R = \varnothing$

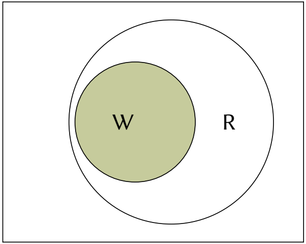
Quantifiers

- one-place predicates can also be seen as sets
- quantifiers express relations between sets
- the universal quantifier expresses a subset relation

(43) Every woman ran.
$= (\forall x)(W(x) \rightarrow R(x)) =$ For every x, if x is female, then x ran.

- in terms of set theory, this means that the the set of women is a subset of the set of individuals that run

(44)



(45) $W \subseteq R$

Quantifiers

- note that the restriction of $\forall$ must be connected with the main predicate of the statement via the connective $\rightarrow$ to render the meaning of *every*
- in contrast, the restriction of $\exists$ must use the connective $\wedge$ to introduce its restriction to correctly model the meaning of *some*

(46) a. $(\forall x)(W(x) \rightarrow R(x))$
= For every x, if x is female, then x ran.

b. $(\exists x)(W(x) \wedge R(x))$
= There is at least one x, such that x is female and x ran.

- compare the following formulas, which do not render the intended meanings (we will come to the precise semantics of predicate logic later)

(47) a. $(\forall x)(W(x) \wedge R(x))$
= For every x, x is female and x ran.
(only true if everyone is a woman and everone runs)

b. $(\exists x)(W(x) \rightarrow R(x))$
= There is at least one x, such that if x is female then x ran.
(even true if there is no woman at all)