Modul 04-006-1001: **WiSe 2023-2024**
Formale Grundlagen (Logik)

Solutions 9

Excercise 1: Proofs

- A statement $P \Rightarrow Q$ claims that Q is a logical consequence of P. One can show that $P \Rightarrow Q$ is true by showing that $P \rightarrow Q$ is a tautology via drawing truth tables.

(1) a. $(p \vee p) \Rightarrow p$:

p	$(p \vee p)$	$((p \vee p) \rightarrow p)$
1	1	1
0	0	1

b. $((\neg p) \rightarrow (\neg q)) \Rightarrow (q \rightarrow p)$:

p	q	$(\neg p)$	$(\neg q)$	$((\neg p) \rightarrow (\neg q))$	$(q \rightarrow p)$	$(((\neg p) \rightarrow (\neg q)) \rightarrow (q \rightarrow p))$
1	1	0	1	1	1	1
0	1	1	0	0	0	1
1	0	0	1	1	1	1
0	0	1	0	0	1	1

c. $(p \vee ((\neg p) \wedge q)) \Rightarrow (p \vee q)$:

p	q	$(\neg p)$	$(p \vee q)$	$((\neg p) \wedge q)$	$(p \vee ((\neg p) \wedge q))$	$((p \vee ((\neg p) \wedge q)) \rightarrow (p \vee q))$
1	1	0	1	0	1	1
0	1	1	1	1	1	1
1	0	0	1	0	1	1
0	0	1	0	0	0	1

d. $(p \rightarrow (q \rightarrow r)) \Rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$:

p	q	r	$(p \rightarrow q)$	$(p \rightarrow r)$	$(q \rightarrow r)$	$(p \rightarrow (q \rightarrow r))$	$((p \rightarrow q) \rightarrow (p \rightarrow r))$	$\ldots$
1	1	1	1	1	1	1	1	1
0	1	1	1	1	1	1	1	1
1	0	1	0	1	1	1	1	1
0	0	1	1	1	1	1	1	1
1	1	0	1	0	0	0	0	1
0	1	0	1	1	0	1	1	1
1	0	0	0	0	1	1	1	1
0	0	0	1	1	1	1	1	1

where "$\ldots$" stands for $((p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)))$

(1) e. $(p \vee q) \Rightarrow (q \vee p)$:

p	q	$(p \vee q)$	$(q \vee p)$	$((p \vee q) \rightarrow (q \vee p))$
1	1	1	1	1
0	1	1	1	1
1	0	1	1	1
0	0	0	0	1

f. $(p \rightarrow q) \Rightarrow ((p \wedge r) \rightarrow (q \wedge r))$:

p	q	r	$(p \wedge r)$	$(q \wedge r)$	$(p \rightarrow q)$	$((p \wedge r) \rightarrow (q \wedge r))$	...
1	1	1	1	1	1	1	1
0	1	1	0	1	1	1	1
1	0	1	1	0	0	0	1
0	0	1	0	0	1	1	1
1	1	0	0	0	1	1	1
0	1	0	0	0	1	1	1
1	0	0	0	0	0	1	1
0	0	0	0	0	1	1	1

where "..." stands for $((p \rightarrow q) \rightarrow ((p \wedge r) \rightarrow (q \wedge r)))$

g. $(\neg p) \Rightarrow (p \rightarrow q)$:

p	q	$(\neg p)$	$(p \rightarrow q)$	$((\neg p) \rightarrow (p \rightarrow q))$
1	1	0	1	1
0	1	1	1	1
1	0	0	0	1
0	0	1	1	1

- Alternatively, one can show that $P \rightarrow Q$ is a tautology by going through a proof by contradiction. Assume that the whole implication is false and lead this assumption to a contradiction (↯). In what follows, the procedure will be illustrated by means of (1-b) and (1-c).

(1-b)

$((\neg p)$	$\rightarrow$	$(\neg q))$	$\rightarrow$	$(q$	$\rightarrow$	$p)$	
			0				
	1				0		
1		0		1		0	↯

If $q = 1$ and $p = 0$, then $(\neg q) = 0$ and $(\neg p) = 1$. But then it is impossible that $((\neg p) \rightarrow (\neg q)) = 1$. Therefore, we have a contradiction.

(1-c)

$(p$	$\vee$	$((\neg p)$	$\wedge$	$q))$	$\rightarrow$	$(p$	$\vee$	$q)$	
					0				
	0						1		
0			0			0		1	
		1		1					↯

If $p = 0$ then $(\neg p) = 1$. But if also $q = 1$, then it is impossible that $((\neg p) \wedge q) = 0$. Thus, there is a contradiction.

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Excercise 2: Conditional proofs

- Give a conditional proof of the validity of each of the following arguments. (Note: In the solutions to the previous exercise sheet, the first argument was proven without auxiliary assumption, the second argument was proven indirectly, namely by contradiction. The present task is to provide a direct conditional proof for both.)

(2)

1.	$(p \rightarrow (\neg q))$	
2.	$(r \rightarrow q)$	
3.	$((\neg r) \rightarrow s)$	
4.	$\mid p$	Auxiliary premise
5.	$\mid (\neg q)$	MP (1, 4)
6.	$\mid (\neg r)$	MT (2, 5)
7.	$\mid s$	MP (3, 6)
$\therefore$	$(p \rightarrow s)$	Cond.Proof (4, 7)

(3)

1.	$((\neg p) \rightarrow q)$	
2.	$(r \rightarrow (s \vee t))$	
3.	$(s \rightarrow (\neg r))$	
4.	$(p \rightarrow (\neg t))$	
5.	$\mid r$	Auxiliary premise
6.	$\mid (s \vee t)$	MP (2, 5)
7.	$\mid (\neg s)$	MT (3, 5)
8.	$\mid t$	Dis.Syll. (6, 7)
9.	$\mid (\neg p)$	MT (4, 8)
10.	$\mid q$	MP (1, 9)
$\therefore$	$(r \rightarrow q)$	Cond.Proof (5, 10)