Modul 04-006-1001: **WiSe 2023-2024**
Formale Grundlagen (Logik)

Solutions 8

Excercise 1: Proofs

- Give formal proofs of the validity of each of the following argument forms by making use of the valid arguments we already gave proofs of (Modus Ponens, Modus Tollens, Disjunctive Syllogism, etc). If a direct proof seems impossible, try a proof by contradition (*reduction ad absurdum*) in the following way: Add to the premises the additional assumption that the consequence of the proof is wrong (i.e., add the negation of the consequence to the premises). Then show that this leads to a contradiction.
- Note: ↯ = contradiction!

(1)

1.	$(p \to q)$	
2.	$(q \to r)$	
3.	$(\neg r)$	
4.	$(\neg q)$	MT (2, 3)
$\therefore$	$(\neg p)$	MT (1, 4)

(2)

1.	p	
2.	$(\neg r)$	
3.	$((p \land (\neg r)) \to q)$	
4.	$(p \land (\neg r))$	Conj. (1, 2)
$\therefore$	q	MP (3, 4)

(3)

1.	$(p \lor q)$	
2.	$(\neg q)$	
3.	$(r \to (\neg p))$	
4.	p	Dis.Syll.
$\therefore$	$(\neg r)$	MT (3, 4)

(4)

1.	$(p \to (\neg q))$	
2.	$(r \to q)$	
3.	$((\neg r) \to s)$	
4.	$(q \to (\neg p))$	Cond. (1)
5.	$(r \to (\neg p))$	Hyp.Syll. (2, 4)
6.	$(p \to (\neg r))$	Cond. (5)
$\therefore$	$(p \to s)$	Hyp.Syll. (3, 6)

(5)

1.	$((\neg p) \lor q)$	
2.	$((\neg q) \land r)$	
3.	$((\neg(p \lor q)) \to s)$	
4.	$((p \lor q) \lor s)$	Cond. (3)
5.	$(\neg q)$	Simpl. (2)
6.	$(\neg p)$	Dis.Syll. (1, 5)
7.	$(p \lor (q \lor s))$	Assoc. (4)
8.	$(q \lor s)$	Dis.Syll (6, 7)
9.	s	Dis.Syll (5, 8)
10.	r	Simpl. (2)
$\therefore$	$(r \land s)$	Conj. (9, 10)

(6)

1.	$(p \lor (q \land r))$	
2.	$(\neg t)$	
3.	$((p \lor q) \to (s \lor t))$	
4.	$(\neg p)$	
5.	$(\neg(p \lor q) \lor (s \lor t))$	Cond. (3)
6.	$((p \lor q) \land (p \lor r))$	Distr. (1)
7.	$(p \lor q)$	Simpl. (6)
8.	$(s \lor t)$	Dis.Syll (5, 7)
9.	s	Dis.Syll (2, 8)
10.	$(q \land r)$	Dis.Syll (1, 4)
11.	r	Simpl. (10)
$\therefore$	$(r \land s)$	Conj. (9, 11)

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(7)

1.	$(p \to q)$	
2.	$(r \to s)$	
3.	$((\neg q) \vee (\neg s))$	
4.	p	
5.	$((t \wedge u) \to r)$	
6.	q	MP (1, 4)
7.	$(\neg s)$	Dis.Syll. (3, 6)
8.	$(\neg r)$	MT (2, 7)
9.	$(\neg (t \wedge u))$	MT (5, 8)
$\therefore$	$((\neg t) \vee (\neg u))$	DeMorgan (9)

(8)

1.	$((p \wedge q) \to (p \to (r \wedge s)))$	
2.	$((p \wedge q) \wedge u)$	
3.	$(p \wedge q)$	Simpl. (2)
4.	$(p \to (r \wedge s))$	MP (1, 3)
5.	p	Simpl. (3)
$\therefore$	$(r \wedge s)$	MP (4, 5)

(9)

1.	$(p \leftrightarrow q)$	
2.	$(\neg p)$	
3.	$((q \wedge (\neg r)) \vee t)$	
4.	$((s \vee t) \to r)$	
5.	$((p \to q) \wedge (q \to p))$	Bicond. (1)
6.	$(q \to p)$	Simpl (5)
7.	$(\neg q)$	MT (2, 6)
8.	$((F \wedge (\neg r)) \vee t)$	(3, 7)
9.	$(F \vee t)$	Ident. (8)
10.	t	Ident. (9)
11.	$((s \vee T) \to r)$	(10, 4)
12.	$(T \to r)$	Ident. (11)
13.	r	MP (12)
$\therefore$	$(r \wedge (\neg q))$	Conj. (12, 13)

(10)

1.	$((\neg p) \to q)$	
2.	$(r \to (s \vee t))$	
3.	$(s \to (\neg r))$	
4.	$(p \to (\neg t))$	
5.	$\neg (r \to q)$	Assumption
6.	$\neg ((\neg r) \vee q)$	Cond. (5)
7.	$(r \wedge (\neg q))$	DeMorgan (6)
8.	r	Simpl. (7)
9.	$(\neg q)$	Simpl. (7)
10.	p	MT (1, 9)
11.	$(\neg t)$	MP (4, 10)
12.	$(s \vee t)$	MP (2, 8)
13.	s	Dis.Syll. (11, 12)
14.	$(\neg r)$	MP (3, 13)
15.	$(r \wedge (\neg r))$ ↯	Conj. (8, 14)
$\therefore$	$(r \to q)$	Negation (5)

(11)

1.	$(p \to (q \wedge r))$	
2.	$(q \to s)$	
3.	$(r \to t)$	
4.	$((s \wedge t) \to (\neg u))$	
5.	u	
6.	p	Assumption
7.	$(q \wedge r)$	MP (1, 6)
8.	q	Simpl. (7)
9.	r	Simpl. (7)
10.	s	MP (2, 8)
11.	t	MP (3, 9)
12.	$(s \wedge t)$	Conj. (10, 11)
13.	$(\neg u)$	MP (4, 12)
14.	$(u \wedge (\neg u))$ ↯	Conj. (5, 13)
$\therefore$	$(\neg p)$	Negation (6)

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(12)			
	1.	p	
	2.	$((p \wedge q) \vee (p \wedge r))$	
	3.	$((p \vee q) \rightarrow (\neg r))$	
	4.	$((\neg(p \vee q)) \vee (\neg r))$	Cond. (3)
	5.	$(((\neg p) \wedge (\neg q)) \vee (\neg r))$	DeMorgan (4)
	6.	$((F \wedge (\neg q)) \vee (\neg r))$	(1, 5)
	7.	$(F \vee (\neg r))$	Ident. (6)
	8.	$(\neg r)$	Ident. (7)
	9.	$((p \wedge q) \vee (p \wedge F))$	(2, 8)
	10.	$((p \wedge q) \vee F)$	Ident. (9)
	11.	$((p \wedge q)$	Ident. (10)
	12.	q	Simpl. (11)
	13.	$(\neg(p \leftrightarrow q))$	Assumption
	14.	$(\neg((p \rightarrow q)) \wedge (q \rightarrow p))$	Bicond. (13)
	15.	$(\neg(p \rightarrow q) \vee (\neg(q \rightarrow p)))$	DeMorgan (14)
	16.	$((\neg((\neg p) \vee q)) \vee (\neg((\neg q) \vee p)))$	Cond. (15)
	17.	$((p \wedge (\neg q)) \vee (q \wedge (\neg p)))$	DeMorgan (16)
	18.	$((p \wedge F) \vee (q \wedge F))$	(1, 12, 17)
	19.	$(F \vee F)$	Ident. (18, 2×)
	20.	$F \lightning$	Ident. (19)
	$\therefore$	$(p \leftrightarrow q)$	Negation (13)