Modul 04-006-1001: **WiSe 2023-2024**
Formale Grundlagen (Logik)

Solutions 5

Excercise 1: Transitivity and connectedness
Let $A = \{1, 2, 3, 4\}$.

- Describe the properties of each relation R_i in A below, of its inverse (R_i^{-1}), and of its complement (R_i') with respect to transitivity and connectedness.

(1)
a. $R_1 = \{\langle 1,1\rangle, \langle 2,1\rangle, \langle 3,4\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle, \langle 4,1\rangle\}$:
non-transitive, non-connected
b. $R_1^{-1} = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 4,3\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle, \langle 1,4\rangle\}$:
non-transitive, non-connected
c. $R_1' = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 4,2\rangle, \langle 4,3\rangle\}$:
non-transitive, connected

(2)
a. $R_2 = \{\langle 3,4\rangle, \langle 1,2\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 1,3\rangle\}$:
transitive, connected
b. $R_2^{-1} = \{\langle 4,3\rangle, \langle 2,1\rangle, \langle 4,1\rangle, \langle 3,2\rangle, \langle 4,2\rangle, \langle 3,1\rangle\}$:
transitive, connected
c. $R_2' = \{\langle 1,1\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \langle 4,1\rangle, \langle 4,2\rangle, \langle 4,3\rangle, \langle 4,4\rangle\}$:
transitive, connected

(3)
a. $R_3 = \{\langle 2,4\rangle, \langle 3,1\rangle, \langle 3,4\rangle, \langle 2,2\rangle, \langle 1,3\rangle, \langle 4,3\rangle, \langle 4,2\rangle\}$:
non-transitive, non-connected
b. $R_3^{-1} = \{\langle 4,2\rangle, \langle 1,3\rangle, \langle 4,3\rangle, \langle 2,2\rangle, \langle 3,1\rangle, \langle 3,4\rangle, \langle 2,4\rangle\}$:
non-transitive, non-connected
c. $R_3' = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,4\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \langle 4,1\rangle, \langle 4,4\rangle\}$:
non-transitive, non-connected

(4)
a. $R_4 = \{\langle 1,1\rangle, \langle 2,4\rangle, \langle 1,3\rangle, \langle 2,2\rangle, \langle 3,1\rangle, \langle 4,4\rangle, \langle 3,3\rangle, \langle 4,2\rangle\}$:
transitive, non-connected
b. $R_4^{-1} = \{\langle 1,1\rangle, \langle 4,2\rangle, \langle 3,1\rangle, \langle 2,2\rangle, \langle 1,3\rangle, \langle 4,4\rangle, \langle 3,3\rangle, \langle 2,4\rangle\}$:
transitive, non-connected
c. $R_4' = \{\langle 1,2\rangle, \langle 1,4\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 3,4\rangle, \langle 4,1\rangle, \langle 4,3\rangle\}$:
intransitive, non-connected

Excercise 2: Partitions

- Is any of the R_i in exercise 1 an equivalence relation (see excercise 5 on sheet 4 for reflexivity and symmetry)? If so, then give the partition that is induced on A.

  R_4 is reflexive, symmetric and transitive, thus an equivalence relation. The partition induced by R_4 is $\{\{1,3\}, \{2,4\}\}$. In contrast, R_1 is anti-symmetric and non-transitive, R_2 is irreflexive and non-transitive, and R_3 is non-reflexive and non-transitive. Therefore, $R_{1,2,3}$ are not equivalence relations.

- Give the equivalence relation that induces the following partition on A: $P = \{\{1\},\{2,3\},\{4\}\}$.

(5) $R_5 = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle\}$

- How many different partitions on A are possible?

  There are 15 possible partitions for a set with 4 members. (But the general formula for computing the number of partitions for a given set is not straightforward.) In the present case ($A = \{1,2,3,4\}$), the partitions are the following:

(6)

a.	$\{\{1\},\{2\},\{3\},\{4\}\}$	h.	$\{\{2,3,4\},\{1\}\}$
b.	$\{\{1,2\},\{3\},\{4\}\}$	i.	$\{\{1,2,3\},\{4\}\}$
c.	$\{\{1,3\},\{2\},\{4\}\}$	j.	$\{\{1,2,4\},\{3\}\}$
d.	$\{\{1,4\},\{3\},\{2\}\}$	k.	$\{\{1,3,4\},\{2\}\}$
e.	$\{\{2,3\},\{1\},\{4\}\}$	l.	$\{\{1,2,3,4\}\}$
f.	$\{\{2,4\},\{1\},\{3\}\}$	m.	$\{\{1,2\},\{3,4\}\}$
g.	$\{\{3,4\},\{1\},\{2\}\}$	n.	$\{\{1,3\},\{2,4\}\}$
		o.	$\{\{1,4\},\{2,3\}\}$

Excercise 3: Orders
Let $A = \{1,2,3,5,6,10,15,30\}$ and let R be the relation in A defined as
$R = \{\langle x,y\rangle \mid x \text{ divides } y \text{ without remainder}\}$

- List the members of R and determine whether they form an order (and if so, whether the order is weak or strong).

  R (as illustrated in (7)) is reflexive, since every number divides itself without remainder. It is anti-symmetric (because except for the reflexive pairs, division without remainder is not symmetric), and it is transitive (because if y is a multiple of x and z is a multiple of y, then z is a multiple of x). Hence, R is a weak order.

(7) $R = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 1,5\rangle, \langle 1,6\rangle, \langle 1,10\rangle, \langle 1,15\rangle, \langle 1,30\rangle, \langle 2,2\rangle, \langle 2,6\rangle, \langle 2,10\rangle, \langle 2,30\rangle, \langle 3,3\rangle, \langle 3,6\rangle, \langle 3,15\rangle, \langle 3,30\rangle, \langle 5,5\rangle, \langle 5,10\rangle, \langle 5,15\rangle, \langle 5,30\rangle, \langle 6,6\rangle, \langle 6,30\rangle, \langle 10,10\rangle, \langle 10,30\rangle, \langle 15,15\rangle, \langle 15,30\rangle, \langle 30,30\rangle\}$

- Do the same for the set $\wp(B)$, where $B = \{a,b,c\}$, and the relation "is a subset of" (relation $\subseteq$).

(8) $\wp(B) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

(9) $\subseteq$ (in $\wp(B)$) $= \{\langle\emptyset,\{a\}\rangle, \langle\emptyset,\{b\}\rangle, \langle\emptyset,\{c\}\rangle, \langle\emptyset,\{a,b\}\rangle, \langle\emptyset,\{a,c\}\rangle, \langle\emptyset,\{b,c\}\rangle, \langle\emptyset,\{a,b,c\}\rangle, \langle\emptyset,\emptyset\rangle,$
$\langle\{a\},\{a\}\rangle, \langle\{a\},\{a,b\}\rangle, \langle\{a\},\{a,c\}\rangle, \langle\{a\},\{a,b,c\}\rangle,$
$\langle\{b\},\{b\}\rangle, \langle\{b\},\{a,b\}\rangle, \langle\{b\},\{b,c\}\rangle, \langle\{b\},\{a,b,c\}\rangle,$
$\langle\{c\},\{c\}\rangle, \langle\{c\},\{a,c\}\rangle, \langle\{c\},\{b,c\}\rangle, \langle\{c\},\{a,b,c\}\rangle,$
$\langle\{a,b\},\{a,b\}\rangle, \langle\{a,b\},\{a,b,c\}\rangle,$
$\langle\{a,c\},\{a,c\}\rangle, \langle\{a,c\},\{a,b,c\}\rangle,$
$\langle\{b,c\},\{b,c\}\rangle, \langle\{b,c\},\{a,b,c\}\rangle,$
$\langle\{a,b,c\},\{a,b,c\}\rangle\}$

$\subseteq$ is reflexive, anti-symmetric and transitive, and therefore defines a weak order.