Modul 04-006-1001: **WiSe 2023-2024**
Formale Grundlagen (Logik)

Solutions 4

Excercise 1: Relations and functions
Let $A = \{a, b, c\}$ and $B = \{1, 2\}$.

- How many distinct relations are there from A to B?
  Each member of the three members of A can relate to only one of the two members of B (two different possibilities), to both, or to none. This means that there are $4^3 = 64$ different relations. Alternatively, one may recall that for a relation R from A to B it holds that $R \subseteq A \times B$. The number of all such relations is $|\wp(A \times B)|$, which, in the present case, is 64.
- How many of these relations are total functions from A to B?
  The total functions are those relations that map every member of A to one member of B (for which there are two possibilities, 1 and 2). This means that there are $2^3 = 8$ such functions.
- How many of these total functions are onto (surjective)?
  A function F from A to B is onto if every member of B is in the range of F. There are only two functions among the 8 total functions from above for which this is not the case: $F_1 = \{\langle a, 1\rangle, \langle b, 1\rangle, \langle c, 1\rangle\}$ and $F_2 = \{\langle a, 2\rangle, \langle b, 2\rangle, \langle c, 2\rangle\}$. This means that 6 of the 8 functions from above are onto.
- How many of these total functions are one-to-one (injective)?
  Since there are more members in A than in B, and since the functions under considerations are total, it follows that none of them can be one-to-one: two members of A always have to be mapped to one and the same member of B.
- Do any of these functions have inverses that are also total functions?
  For the inverse F^{-1} of a function F to also be a function, F must be one-to-one. Since this is not the case for any of the functions under consideration (see previous question), their inverses are not functions.
- Answer the same questions for all relations from B to A.
  Every of the two members of B can relate in 8 different ways to members of A (it can relate to a, to b, to c, to a and b, to a and c, etc., or to nothing, which makes 8 possibilities). The total number of relations is therefore $8^2 = 64$. The number of total functions among these relations is $3^2 = 9$. None of them are onto (since there are more members in A than in B). There are three functions that are not one-to-one ($F_1 = \{\langle 1, a\rangle, \langle 2, a\rangle\}$, $F_2 = \{\langle 1, b\rangle, \langle 2, b\rangle\}$, $F_3 = \{\langle 1, c\rangle, \langle 2, c\rangle\}$), which leaves 6 of the 9 functions being one-to-one. Since none of the functions are onto, this means that none of them has an inverse that also is a *total* function.

Excercise 2: Composition

- Let R_1 and R_2 be the following two relations in $A = \{1, 2, 3, 4\}$:

(1) a. $R_1 = \{\langle 1, 1\rangle, \langle 2, 1\rangle, \langle 3, 4\rangle, \langle 2, 2\rangle, \langle 3, 3\rangle, \langle 4, 4\rangle, \langle 4, 1\rangle\}$
b. $R_2 = \{\langle 3, 4\rangle, \langle 1, 2\rangle, \langle 1, 4\rangle, \langle 2, 3\rangle, \langle 2, 4\rangle, \langle 1, 3\rangle\}$

- Form the composites $R_2 \circ R_1$ and $R_1 \circ R_2$. Are they equal?

(2) a. $R_2 \circ R_1 = \{\langle 1, 2\rangle, \langle 1, 3\rangle, \langle 1, 4\rangle, \langle 2, 2\rangle, \langle 2, 3\rangle, \langle 2, 4\rangle, \langle 3, 4\rangle, \langle 4, 2\rangle, \langle 4, 3\rangle, \langle 4, 4\rangle\}$

b. $R_1 \circ R_2 = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 3,1\rangle, \langle 3,4\rangle\}$
c. $R_2 \circ R_1 \neq R_1 \circ R_2$

- Show that $R_1^{-1} \circ R_1 \neq id_A$ and that $R_2^{-1} \circ R_2 \not\subseteq id_A$.

(3) a. $R_1^{-1} \circ R_1 = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,4\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 2,4\rangle, \langle 3,3\rangle, \langle 3,4\rangle, \langle 4,1\rangle, \langle 4,2\rangle, \langle 4,3\rangle, \langle 4,4\rangle\} \neq id_A = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle\}$
b. $R_2^{-1} \circ R_2 = \{\langle 3,3\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 1,1\rangle, \langle 1,3\rangle, \langle 1,2\rangle, \langle 2,2\rangle, \langle 2,1\rangle, \langle 2,3\rangle\} \not\subseteq id_A$

Excercise 3: Composition and the inverse

- Let F and G in (4-a,b) be functions from $A = \{a, b, c\}$ to $B = \{1, 2, 3, 4\}$ and from $C = \{1, 2, 3, 4\}$ to $D = \{p, q, r\}$, respectively. Show that $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$.

(4) a. $F = \{\langle a,1\rangle, \langle b,3\rangle, \langle c,3\rangle\}$
b. $G = \{\langle 1,p\rangle, \langle 2,q\rangle, \langle 3,q\rangle, \langle 4,r\rangle\}$

(5) a. $G \circ F = \{\langle a,p\rangle, \langle b,q\rangle, \langle c,q\rangle\}$
b. $(G \circ F)^{-1} = \{\langle p,a\rangle, \langle q,b\rangle, \langle q,c\rangle\}$

(6) a. $F^{-1} = \{\langle 1,a\rangle, \langle 3,b\rangle, \langle 3,c\rangle\}$
b. $G^{-1} = \{\langle p,1\rangle, \langle q,2\rangle, \langle q,3\rangle, \langle r,4\rangle\}$
c. $F^{-1} \circ G^{-1} = \{\langle p,a\rangle, \langle q,b\rangle, \langle q,c\rangle\} =$ (5-b)

Excercise 4: Reflexivity and symmetry

- Give the status for the two relations "is a child of" and "is a brother of" (in the set of human beings) with respect to reflexivity and symmetry. Only name the strongest relevant property if the relation in question has more than one (e.g., a relation that is irreflexive is also non-reflexive, but not vice versa: irreflexivity is stronger than non-reflexivity).

(7) a. "is a child of": irreflexive, asymmetric
b. "is a brother of": irreflexive, non-symmetric

Excercise 5: More reflexivity and symmetry

- Let $A = \{1, 2, 3, 4\}$.
- Describe the properties of each relation R_i in A below, of its inverse (R_i^{-1}), and of its complement (R_i') with respect to reflexivity and symmetry.

(8) a. $R_1 = \{\langle 1,1\rangle, \langle 2,1\rangle, \langle 3,4\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle, \langle 4,1\rangle\}$:
reflexive, anti-symmetric
b. $R_1^{-1} = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 4,3\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle, \langle 1,4\rangle\}$:
reflexive, anti-symmetric
c. $R_1' = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 4,2\rangle, \langle 4,3\rangle\}$:
irreflexive, non-symmetric

(9) a. $R_2 = \{\langle 3,4\rangle, \langle 1,2\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,4\rangle, \langle 1,3\rangle\}$:
irreflexive, asymmetric
b. $R_2^{-1} = \{\langle 4,3\rangle, \langle 2,1\rangle, \langle 4,1\rangle, \langle 3,2\rangle, \langle 4,2\rangle, \langle 3,1\rangle\}$:
irreflexive, asymmetric

c. $R_2' = \{\langle 1,1\rangle, \langle 2,1\rangle, \langle 2,2\rangle, \langle 3,1\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \langle 4,1\rangle, \langle 4,2\rangle, \langle 4,3\rangle, \langle 4,4\rangle\}$:
reflexive, anti-symmetric

(10) a. $R_3 = \{\langle 2,4\rangle, \langle 3,1\rangle, \langle 3,4\rangle, \langle 2,2\rangle, \langle 1,3\rangle, \langle 4,3\rangle, \langle 4,2\rangle\}$:
non-reflexive, symmetric

b. $R_3^{-1} = \{\langle 4,2\rangle, \langle 1,3\rangle, \langle 4,3\rangle, \langle 2,2\rangle, \langle 3,1\rangle, \langle 3,4\rangle, \langle 2,4\rangle\}$:
non-reflexive, symmetric

c. $R_3' = \{\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,4\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 3,3\rangle, \langle 4,1\rangle, \langle 4,4\rangle\}$:
non-reflexive, symmetric

(11) a. $R_4 = \{\langle 1,1\rangle, \langle 2,4\rangle, \langle 1,3\rangle, \langle 2,2\rangle, \langle 3,1\rangle, \langle 4,4\rangle, \langle 3,3\rangle, \langle 4,2\rangle\}$:
reflexive, symmetric

b. $R_4^{-1} = \{\langle 1,1\rangle, \langle 4,2\rangle, \langle 3,1\rangle, \langle 2,2\rangle, \langle 1,3\rangle, \langle 4,4\rangle, \langle 3,3\rangle, \langle 2,4\rangle\}$:
reflexive, symmetric

c. $R_4' = \{\langle 1,2\rangle, \langle 1,4\rangle, \langle 2,1\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 3,4\rangle, \langle 4,1\rangle, \langle 4,3\rangle\}$:
irreflexive, symmetric