Modul 04-006-1001: **WiSe 2023-2024**
Formale Grundlagen (Logik)

Solutions 3

Excercise 1: Set operations and membership

(1) a. $A = \{a, b, c\}$
b. $B = \{c, d\}$
c. $C = \{d, e, f\}$

(2) a. $A \cup B = \{a, b, c, d\}$
b. $A \cap B = \{c\}$
c. $A \cup (B \cap C) = \{a, b, c, d\}$
d. $C \cup A = \{a, b, c, d, e, f\}$
e. $B \cup \emptyset = B$
f. $A \cap (B \cap C) = \emptyset$
g. $A - B = \{a, b\}$

- Is a a member of $\{A, B\}$? No, because $\{A, B\}$ only has A and B as members.
- Is a a member of $A \cup B$? Yes, see (2-a).

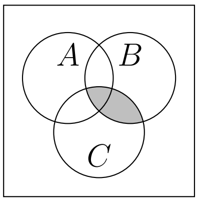
Excercise 2: Set theoretic equations
- Show by using the set-theoretic equalities that were introduced (idempotent laws, commutative laws, etc.) that the following holds for any sets A and B: $A \cap (B - A) = \emptyset$.

(3) a. $A \cap (B - A) =$ (Compl.)
b. $A \cap (B \cap A') =$ (Commut.)
c. $A \cap (A' \cap B) =$ (Assoc.)
d. $(A \cap A') \cap B =$ (Compl.)
e. $\emptyset \cap B =$ (Ident.)
f. $\emptyset$

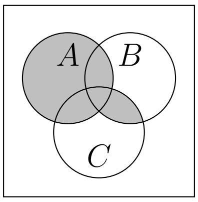
Excercise 3: Venn diagramms and distributive law
- Show by means of Venn diagramms that the equation in (4) holds.

(4) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

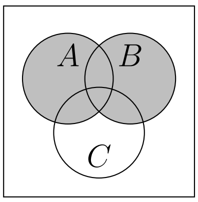
(5) $B \cap C$:



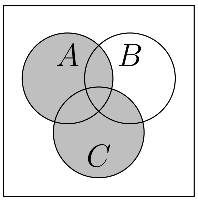
$A \cup (B \cap C)$:



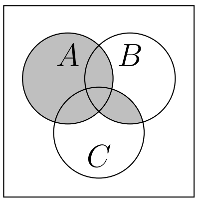
(6) $A \cup B$:



$A \cup C$:



$(A \cup B) \cap (A \cup C)$:

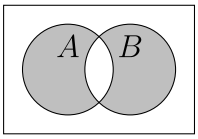


Excercise 4: Symmetric difference

- Draw the Venn diagramm for the symmetric difference of two sets (7-a).
- Show that (7-b) holds by making reference to set theoretic equalities. Verify that the Venn diagramm for $(A-B)\cup(B-A)$ is the same as the diagramm for $A+B$.
- Show that for all sets A and B: $A+B=B+A$.

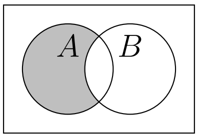
(7) a. $A+B =_{def} (A\cup B)-(A\cap B)$
b. $A+B=(A-B)\cup(B-A)$

(8)

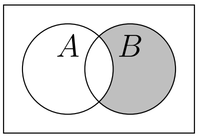


(9) a. $(A-B)\cup(B-A) =$ (Compl. 2×)
b. $(A\cap B')\cup(B\cap A') =$ (Distr.)
c. $((A\cap B')\cup B)\cap((A\cap B')\cup A') =$ (Distr. 2×)
d. $((A\cup B)\cap(B'\cup B))\cap((A\cup A')\cap(B'\cup A')) =$ (Compl. 2×)
e. $(A\cup B)\cap U)\cap(U\cap(B'\cup A')) =$ (Ident. 2×)
f. $(A\cup B)\cap(B'\cup A') =$ (DeMorgan)
g. $(A\cup B)\cap(B\cap A)' =$ (Compl.)
h. $(A\cup B)-(B\cap A) =$ (Commut.)
i. $(A\cup B)-(A\cap B) =$ (Def.)
j. $A+B$

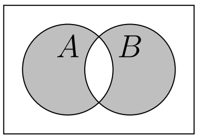
(10) $A-B$:



$B-A$:



$(A-B)\cup(B-A)$:



(11) a. $A+B =$ (Def.)
b. $(A\cup B)-(A\cap B) =$ (Commut. 2×)
c. $(B\cup A)-(B\cap A) =$ (Def.)
d. $B+A$

Excercise 5: More on symmetric difference

(12) a. $A+A=\emptyset$
b. $A+U=A'$
c. $A+\emptyset=A$
d. $A+B$ (with $A\subseteq B$) $= B-A$
e. $A+B$ (with $A\cap B=\emptyset$) $= A\cup B$

- Show that the statements in (13-a,b) are correct.

(13) a. $((A-B)+(B-A)) = A+B$
b. $(A+B)\subseteq B$ iff $A\subseteq B$

(14) a. $((A - B) + (B - A)) =$ (Def.)
b. $((A - B) \cup (B - A)) - ((A - B) \cap (B - A)) =$ (Compl. 4×)
c. $((A \cap B') \cup (B \cap A')) - ((A \cap B') \cap (B \cap A')) =$ (Distr. 2×)
d. $(((A \cap B') \cup B) \cap ((A \cap B') \cup A')) -$
$(((A \cap B') \cap B) \cap ((A \cap B') \cap A')) =$ (Distr. 2×)
e. $((((A \cup B) \cap (B' \cup B)) \cap ((A \cup A') \cap (B' \cup A'))) -$
$(((A \cap B') \cap B) \cap ((A \cap B') \cap A')) =$ (Commut. + Assoc. 2×)
f. $((((A \cup B) \cap (B' \cup B)) \cap ((A \cup A') \cap (B' \cup A'))) -$
$(((A \cap (B' \cap B)) \cap (B' \cap (A \cap A'))) =$ (Compl. 2×)
g. $(((A \cup B) \cap U) \cap (U \cap (B' \cup A'))) -$
$(((A \cap (B' \cap B)) \cap (B' \cap (A \cap A'))) =$ (Compl. 2×)
h. $(((A \cup B) \cap U) \cap (U \cap (B' \cup A'))) - (((A \cap \emptyset) \cap (B' \cap \emptyset)) =$ (Ident. 2×)
i. $(((A \cup B) \cap U) \cap (U \cap (B' \cup A'))) - (\emptyset \cap \emptyset) =$ (Ident.)
j. $(((A \cup B) \cap U) \cap (U \cap (B' \cup A'))) - \emptyset =$ (Ident. 2×)
k. $((A \cup B) \cap (B' \cup A')) - \emptyset =$ (Set difference)
l. $(A \cup B) \cap (B' \cup A') =$ (Commut.)
m. $(A \cup B) \cap (A' \cup B') =$ (DeMorgan)
n. $(A \cup B) \cap (A \cap B)' =$ (Compl.)
o. $(A \cup B) - (A \cap B) =$ (Def.)
p. $A + B$

(15) a. $(A + B) \subseteq B$ iff (Consist.)
b. $(A + B) \cup B = B$ iff (Def.)
c. $((A \cup B) - (A \cap B)) \cup B = B$ iff (Compl.)
d. $((A \cup B) \cap (A \cap B)') \cup B = B$ iff (DeMorgan)
e. $((A \cup B) \cap (A' \cup B')) \cup B = B$ iff (Distr.)
f. $((A \cup B) \cup B) \cap ((A' \cup B') \cup B) = B$ iff (Assoc. 2×)
g. $(A \cup (B \cup B)) \cap (A' \cup (B' \cup B)) = B$ iff (Compl. + Idempot.)
h. $(A \cup B) \cap (A' \cup U) = B$ iff (Ident.)
i. $(A \cup B) \cap U = B$ iff (Ident.)
j. $(A \cup B) = B$ iff (Consist.)
k. $A \subseteq B$

Excercise 6: Carthesian products and relations

- Given are the sets $A = \{b, c\}$ and $B = \{2, 3\}$.

(16) a. $A \times B = \{\langle b, 2\rangle, \langle b, 3\rangle, \langle c, 2\rangle, \langle c, 3\rangle\}$
b. $B \times A = \{\langle 2, b\rangle, \langle 2, c\rangle, \langle 3, b\rangle, \langle 3, c\rangle\}$
c. $A \times A = \{\langle b, b\rangle, \langle b, c\rangle, \langle c, b\rangle, \langle c, c\rangle\}$
d. $(A \cup B) \times B = \{\langle b, 2\rangle, \langle b, 3\rangle, \langle 2, 2\rangle, \langle 2, 3\rangle, \langle c, 2\rangle, \langle c, 3\rangle, \langle 3, 2\rangle, \langle 3, 3\rangle\}$
e. $(A \cap B) \times B = \emptyset$
f. $(A - B) \times (B - A) = A \times B$

- Consider now the following relation from A to $(A \cup B)$:
  $R = \{\langle b, b\rangle, \langle b, 2\rangle, \langle c, 2\rangle, \langle c, 3\rangle\}$

(17)
a. Range(R): $\{b, 2, 3\}$; Domain(R): $\{b, c\}$
b. $R' = \{\langle b, 3\rangle, \langle b, c\rangle, \langle c, b\rangle, \langle c, c\rangle\}$
c. $R^{-1} = \{\langle b, b\rangle, \langle 2, b\rangle, \langle 2, c\rangle, \langle 3, c\rangle\}$
d. $(R')^{-1} = \{\langle 3, b\rangle, \langle c, b\rangle, \langle b, c\rangle, \langle c, c\rangle\}$
e. $(R^{-1})' = \{\langle b, c\rangle, \langle b, 2\rangle, \langle b, 3\rangle, \langle c, c\rangle, \langle c, b\rangle, \langle c, 2\rangle, \langle c, 3\rangle\}$