Modul 04-006-1001: **WiSe 2023-2024**
Formale Grundlagen (Logik)

Solutions 10

Excercise 1: Translation into English

- Translate the following English sentences into predicate logic. Choose your own variables and predicate letters, giving the key.

(1)
a. Everything is black or white.
$(\forall x)(B(x) \vee W(x))$
$B(x) =$ 'x is black'; $W(x) =$ 'x is white'

b. A dog is a quadruped.
$(\forall x)(D(x) \rightarrow Q(x))$
$D(x) =$ 'x is a dog'; $Q(x) =$ 'x is quadruped'

c. Everybody loves somebody.
$(\forall x)(\exists y)(L(x,y))$
$L(x,y) =$ 'x loves y'

d. There is someone whom everyone loves.
$(\exists y)(\forall x)(L(x,y))$
$L(x,y) =$ 'x loves y'

e. No one loves himself, unless it is John.
$(\forall x)(L(x,x) \rightarrow (x = j))$
$L(x,y) =$ 'x loves y'; $j =$ John

f. People who live in Leipzig love it.
$(\forall x)((P(x) \wedge M(x,l)) \rightarrow L(x,l))$
$P(x) =$ 'x is a person'; $M(x,y) =$ 'x lives in y'; $L(x,y) =$ 'x loves y'; l = Leipzig

g. If someone does not love Leipzig, he does not know it.
$(\forall x)((\neg L(x,l)) \rightarrow (\neg K(x,l)))$
$L(x,y) =$ 'x loves y'; l = Leipzig; $K(x,y) =$ 'x knows y'

h. Give him a finger, and he takes the whole hand.
$(\forall x)(\forall y)((F(y,x) \wedge G(x,y,g)) \rightarrow (\exists z)(H(z,y) \wedge I(g,z,x)))$
$F(x,y) =$ 'x is a finger of y'; $G(x,y,z) =$ 'x gives y to z'; $H(x,y) =$ 'x is the hand of which y is a finger'; $I(x,y,z) =$ 'x takes y from z'; $g =$ 'he'

i. If someone is noisy, he annoys everybody.
$((\exists x)N(x) \rightarrow (\forall y)A(y))$;
$N(x) =$ 'x is noisy'; $A(x) =$ 'x is annoyed'

j. Although no one made noise, John was annoyed.
$((\neg(\exists x)N(x)) \wedge A(j))$
$N(x) =$ 'x is noisy'; $A(x) =$ 'x is annoyed'; $j =$ 'John'

k. Only drunk drivers under 18 cause bad accidents.
$(\forall x)(A(x) \rightarrow B(x))$
$A(x) =$ 'x causes a bad accident'; $B(x) =$ 'x is a drunk driver under 18'

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Excercise 2: Bound vs. free variables

- For each of the expressions below, state whether the statement is open (i.e., contains unbound variables). Name the free variables (i.e. the variables that are unbound).

(2) a. $(\forall x)(P(x) \vee Q(x,y))$
Open statement: x is bound by $\forall x$; y is free.
b. $(\forall y)(Q(x) \rightarrow (\forall z)P(y,z))$
Open statement: y is bound by $\forall y$; z is bound by $\forall z$; x is free.
c. $(\forall x)(P(x) \rightarrow (\exists y)(Q(y) \rightarrow (\forall z)R(y,z)))$
All variables are bound: x by $\forall x$, y by $\exists y$ and z by $\forall z$.

Excercise 3: Predicate logic; models and quantifiers

- Evaluate the truth-values of the following expressions based on model M:

(3) $M = \langle D, I \rangle$, where:
a. $D = \{$*Sokrates, Aristotle, Plato, Michelangelo, Bach, Tarski*$\}$
b. $I =$

term	value	predicate	value
s	*Sokrates*	H	$\{$*Sokrates, Aristotle, Plato*$\}$
m	*Michelangelo*	M	$\{$*Sokrates, Aristotle, Plato, Michelangelo, Bach, Tarski*$\}$
b	*Bach*		
t	*Tarski*	L	$\{\langle$*Sokrates, Sokrates*$\rangle$, $\langle$*Sokrates, Aristotle*$\rangle$, $\langle$*Michelangelo, Bach*$\rangle$, $\langle$*Bach, Michelangelo*$\rangle$, $\langle$*Tarski, Plato*$\rangle$, $\langle$*Plato, Michelangelo*$\rangle$, $\langle$*Aristotle, Tarski*$\rangle\}$
a	*Aristotle*		
p	Plato		

(4) a. $[\![(\exists y)(\forall x)L(x,y)]\!]^M =$ FALSE

Reasoning: The statement above is TRUE iff there is at least one individual y in D, which appears as the second member of an ordered pair in L, for every member in D. We assign a temporary value in D to y and see whether, for that value of y, $[\![(\forall y)L(x,y)]\!]^M$ is TRUE. Given that the outermost quantifier is an existential quantifier, we just need to find one possible value of y that makes $[\![(\forall y)L(x,y)]\!]^M$ TRUE; then we would have shown that the entire expression is TRUE. Let us start by temporarily giving y the value *Socrates*; i.e. $[\![y]\!]^M =$ *Socrates*. We see that, there is only one ordered pair, namely $\langle$*Socrates, Socrates*$\rangle$, in L in which y occurs as the second member of L. In other words, when $[\![x]\!]^M$ =*Socrates*, then y (denoting *Socrates*) occurs as the second member of the ordered pair $\langle x, y\rangle$ in L; for all other possible values of x in D, $[\![y]\!]^M =$ *Socrates* does not occur as the second member of the ordered pair. We now go back and assign y the temporary value *Aristotle*. For this value of y, there are no values of x such that y is the second member of the ordered pair in L. We thus cycle through all possible values of y in D. In doing so, we see that there is no value we can assign to y such that the expression $[\![(\forall y)L(x,y)]\!]^M$is TRUE. Thus, $[\![(\exists y)(\forall x)L(x,y)]\!]^M =$ FALSE.

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(4) b. $[\![(\forall x)\neg(\exists y)L(x,y)]\!]^M =$ FALSE

Reasoning: This statement is TRUE iff for every individual x in the domain D, there is no individual y in D such that x is the first member of an ordered pair in L and y the second member. This means that as soon as we find a value for y such that y is the second member of an ordered pair (and x is any member in D), then the entire statement is FALSE. Here again, we start with the outermost quantifier and temporarily assign values to x in D. Let $[\![x]\!]^M =$ *Socrates*; we now turn to y and see if we can find a value for y that makes y the second member of an ordered pair. As it turns out, when $[\![y]\!]^M =$ *Socrates*, it is the second member of the ordered pair in L, for $[\![x]\!]^M =$ *Socrates*. Thus $[\![(\forall x)\neg(\exists y)L(x,y)]\!]^M =$ FALSE

(4) c. $[\![((\exists z)M(z) \wedge (\forall y)(H(y) \rightarrow L(y,b)))]\!]^M =$ FALSE

Reasoning: This statement is TRUE iff there is at least one individual z in the domain D which is a member of M and for every individual y in D, if y is a member of H, then it is the first member of an ordered pair in L when the second member of that ordered pair is *Bach*. In order for the whole statement to be TRUE, each of the conjuncts must be TRUE, so we test them one by one. First we test the truth value of $[\![(\exists z)M(z)]\!]^M$. We start with $[\![x]\!]^M =$ *Tarski*. Is *Tarski* a member of M? Yes, it is. Thus, when x denotes *Tarski*, it makes M TRUE. I.e. there is at least one member of D that is also a member of M. Thus: $[\![(\exists z)M(z)]\!]^M = 1$. Now we determine the truth-value of $[\![(\forall y)(H(y) \rightarrow L(y,b))]\!]^M$. Is there a value of y that makes the antecedent $[\![H(y)]\!]^M$ be TRUE and the consequent $[\![L(y,b)]\!]^M$ be FALSE? Yes, there is. In fact, when y denotes either *Socrates*, *Aristotle* or *Plato*, it is a member of H but it is not the second member of an ordered pair in L where the second value is *Bach*, thus for these values $[\![H(y)]\!]^M$ is TRUE M and $[\![L(y,b)]\!]^M$ is FALSE. Thus, $[\![(\forall y)(H(y) \rightarrow L(y,b))]\!]^M =$ FALSE. Thus the entire expression $[\![((\exists z)M(z) \wedge (\forall y)(H(y) \rightarrow L(y,b)))]\!]^M$ is also FALSE.

Exercises 4:

- Given the equivalences in (5), prove the equivalence between (6-a,b). Give the names of the laws of logic that you make reference to in your proof.

(5) a. $(\neg(\forall x)(P(x))) \Leftrightarrow (\exists x)\neg(P(x))$
b. $(\neg(\exists x)(P(x))) \Leftrightarrow (\forall x)\neg(P(x))$

(6) a. Kein Kind fährt nicht nach Rom.
$(\neg(\exists x)(K(x) \wedge (\neg F(x,r))))$
b. Alle Kinder fahren nach Rom.
$(\forall x)(K(x) \rightarrow F(x,r))$

(7) a. $(\neg(\exists x)(K(x) \wedge (\neg F(x,r)))) \Leftrightarrow$ ((5-b))
b. $(\forall x)(\neg(K(x) \wedge (\neg F(x,r)))) \Leftrightarrow$ (DeMorgan's Law)
c. $(\forall x)((\neg K(x)) \vee (\neg(\neg F(x,r)))) \Leftrightarrow$ (Complement Law)
d. $(\forall x)((\neg K(x)) \vee F(x,r)) \Leftrightarrow$ (Conditional Law)
e. $(\forall x)(K(x) \rightarrow F(x,r))$