Characterizations of weighted and equal division values

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Abstract

New and recent axioms for cooperative games with transferable utilities are introduced. The *non-negative player axiom* requires to assign a non-negative payoff to a player that belongs to coalitions with non-negative worth only. The axiom of *addition invariance on bi-partitions* requires that the payoff vector recommended by a value should not be affected by an identical change in worth of both a coalition and the complementary coalition. The *nullified solidarity axiom* requires that if a player who becomes null weakly loses (gains) from such a change, then every other player should weakly lose (gain) too. We study the consequence of imposing some of these axioms in addition to some classical axioms. It turns out that the resulting values or set of values have all in common to split efficiently the worth achieved by the grand coalition according to an exogenously given weight vector. As a result, we also obtain new characterizations of the equal division value.

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1. Introduction

The axioms employed to design values in cooperative game theory with transferable utilities can be divided up into punctual and relational axioms (see Thomson, 2012). A punctual axiom applies to each game separately and a relational axiom relates payoff vectors of games that are related in a certain way. This article introduces one new punctual axiom and one new relational axiom.

The well-established *null player axiom* and *nullifying player axiom* are punctual axioms. The former axiom recommends to assign a zero payoff to a null player, *i.e.*, a player with zero contributions to coalitions. The latter axiom enforces a zero payoff to a nullifying player, *i.e.*, a player belonging to coalitions with zero worth only. These two axioms play important roles since they

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enable to distinguish the Shapley value (Shapley, 1953) from the equal division value, two values built on opposite equity principles (see van den Brink, 2007).

There exist few more axioms that rest on the null and nullifying players, or on variants of these types of players. Three examples are *the null player out axiom* (Derks and Haller, 1999), *the null player in a productive environment axiom* (Casajus and Huettner, 2013) and *the nullified solidarity axiom* (Béal et al., 2014). The first one is a relational axiom stating that removing a null player from a game does not affect the payoff of the remaining players. The second one is a punctual axiom that specifies to assign a non-negative payoff to a null player if the grand coalition has a non-negative worth. The third one compares a game before and after a specified player becomes null, *i.e.*, the worth of a coalition in this new game is the worth of the coalition without the specified player in the original game. *Nullified solidarity* is a relational axiom requiring that if the specified player loses from such a change, then every other player should lose too, albeit the magnitudes of these losses can vary.

In this article, we call upon *the null player in a productive environment axiom* and *nullified solidarity*, and introduce a variant of *the nullifying player axiom* called *the non-negative player axiom*. This axiom is a punctual axiom that requires to assign a non-negative payoff to a non-negative player, *i.e.*, a player belonging to coalitions with non-negative worth only. Any nullifying player is also a non-negative player, but the nullifying player and non-negative player axioms are not related to each other, *i.e.*, neither axiom implies the other.

The vast category of relational axioms includes as a subclass the axioms of invariance. Such axioms specify either the same payoff vector or the same payoff for some specific players across games that are related in certain ways. Besides *the null player out axiom*, a well-known example of axiom of invariance is the axiom of *marginality* (Young, 1985), which requires to attribute the same payoff to a player in two games where his marginal contributions to coalitions are identical. Further axioms of invariance are discussed by Béal et al. (2015a,c).

We introduce a new axiom of invariance relying on the idea of bi-partitions, which dates back to von Neumann and Morgenstern (1953).[1] More specifically, our axiom of *addition invariance on bi-partitions* states that the chosen payoff vector should not be affected by an identical change in worth of both a coalition and the complementary coalition. We show that this axiom is equivalent to *self-duality* if one restricts to the domain of additive values, which means that it is satisfied by a lot of well-known values.

We study the consequence of imposing some of the aforementioned axioms in addition to some classical axioms such as *efficiency*, *additivity*, *linearity*, or *the equal treatment axiom*. It turns out that the resulting values or classes of values have all in common to split efficiently the worth achieved by the grand coalition according to an exogenously given weight vector summing up to unity. We refer to the weighted division values when the weight vector can contain negative coordinates, and to the positively weighted division values for the subclass of weighted division values with non-negative weights. Naturally, the equal division value is the unique weighted division value with identical weights. All in all, the article contains ten characterizations of such values or classes of values. To the best of our knowledge, the only similar articles in cooperative game theory are due to van den Brink (2009) who obtains a characterization of the class of all weighted division values by imposing the axiom of *collusion neutrality* (see Haller, 1994) in addition to *linearity* and *efficiency*, and Béal et al. (2015c), who characterize positively weighted division values (resp.

[1]Bi-partitions are also used by Eisenman (1967) and Evans (1996) in studies on the Shapley value.

positively weighted surplus division values) by means of *efficiency*, *linearity* and the axiom of *invariance from player deletion in presence of a nullifying* (resp. *dummifying*) *player*.[2]

The weighted division values constitute an interesting class of values for at least two reasons. Firstly, although the requirement to treat substitute players equally appears to be natural in many situations, it is desirable to have the option of treating substitute players differently in order to reflect exogenous characteristics, such as income or health status. This can be achieved by incorporating exogenous weights into the construction of a value. Weighted values have been popularized by Kalai and Samet (1987) who study the weighted Shapley values. In a sense, the weighted division values generalize the equal division value as the weighted Shapley values generalize the Shapley value. Secondly, proportional division methods are very often employed in a lot of applications such as claim problems, cost allocation problems, insurance, law and so on. We refer to Tijs and Driessen (1986), Lemaire (1991), Balinski and Young (2001), and Thomson (2003) for rich surveys, and to Chun (1988), Moulin (1987), and Thomson (2013) for proportional division methods that rest on exogenously given weights.

Our study exhibits further interesting aspects. From a theoretical point of view, the axiomatic characterizations of the equal division value always rest on at least one of the classical axioms of *efficiency*, *the equal treatment axiom*, or *linearity/additivity*. Some of our results avoid to use some of or all these axioms. As an example, Theorem 2 proves that the equal division value is characterized by *addition invariance on bi-partitions*, *the nullifying player axiom*, and *weak covariance*, where this last axiom is a weak version of *covariance* in the sense that the added additive game is symmetric. Moreover, two of our characterizations of the positively weighted division values give insight into the role of *the equal treatment axiom* in the characterizations of the equal division value. While the role of *the equal treatment axiom* is obvious in these two characterizations of the equal division value, it is more difficult to grasp in the characterization provided by van den Brink (2007).

The rest of the article is organized as follows. Section 2 presents the basic material about cooperative games with transferable utilities. Section 3 introduces the axiom of *addition invariance on bi-partitions*, and contains all the results in which this axiom is invoked. Section 4 defines *the non-negative player* and *nullified solidarity axioms*, and offers the results mobilizing these axioms. A comparison with the main result in van den Brink (2007) is provided in Section 5. Section 6 concludes. Finally, the logical independence of the axioms used in each of our characterizations is demonstrated in the appendix.

2. Basic definitions and notations

Let $N = \{1, \dots, n\}$, $n \in \mathbb{N}$, be the set of players, which is fixed throughout the article. A TU-game on N, or simply a game, is given by the **coalition function** $v \in \mathbb{V} := \{f : 2^N \longrightarrow \mathbb{R} \mid f(\emptyset) = 0\}$. Subsets of N are called coalitions. We write i instead of $\{i\}$ for each singleton coalition. The size of a coalition S is denoted by its lower-case version s; and $v(S)$ is called the worth of coalition S.

For all $c \in \mathbb{R}$, the **symmetric additive game** induced by c is denoted by $\mathbf{c}$ and is given by $\mathbf{c}(S) = s \cdot c$ for all $S \subseteq N$. The particular case $c = 0$ gives rise to the **null game** $\mathbf{0}$ given by $\mathbf{0}(S) = 0$ for all $S \subseteq N$. For $v, w \in \mathbb{V}$ and $c \in \mathbb{R}$, the coalition functions $v + w$ and $c \cdot v$ are given

[2] Dummifying players are introduced in Casajus and Huettner (2014).

by $(v+w)(S) = v(S) + w(S)$ and $(c \cdot v)(S) = c \cdot v(S)$ for all $S \subseteq N$. For $\emptyset \subsetneq T \subseteq N$, the game e_T given by $e_T(S) = 1$ if $S = T$ and $e_T(S) = 0$ for $S \neq T$ is called the **standard game** induced by T. Obviously, any $v \in \mathbb{V}$ admits a unique representation in terms of standard games:

$$v = \sum_{\emptyset \subsetneq T \subseteq N} v(T) \cdot e_T. \tag{1}$$

For $\emptyset \subsetneq T \subseteq N$, the game u_T given by $u_T(S) = 1$ if $S \supseteq T$ and $u_T(S) = 0$ if $S \not\supseteq T$ is called the **unanimity game** induced by T. The **dual** of a game v is the game v^D given by $v^D(S) = v(N) - v(N \setminus S)$ for all $S \subseteq N$.

Player $i \in N$ is **null** in $v \in \mathbb{V}$ if $v(S) = v(S \setminus i)$ for all $S \ni i$. Player $i \in N$ is **nullifying** in $v \in \mathbb{V}$ if $v(S) = 0$ for all $S \ni i$. Player $i \in N$ is **non-negative** in $v \in \mathbb{V}$ if $v(S) \geq 0$ for all $S \ni i$. Two distinct players $i, j \in N$ are **substitutes** in $v \in \mathbb{V}$ if $v(S \cup i) = v(S \cup j)$ for every $S \subseteq N \setminus \{i, j\}$.

A **value** is a function φ that assigns a payoff vector $\varphi(v) \in \mathbb{R}^n$ to any $v \in \mathbb{V}$. We consider the following values. Let $\Delta^n := \left\{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = 1\right\}$ and $\Delta^n_+ := \Delta^n \cap \mathbb{R}^n_+$. For $\omega \in \Delta^n$, the **ω-weighted division value** WD^ω is given by

$$\mathrm{WD}^\omega_i(v) = \omega_i \cdot v(N) \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

The class of all weighted division values is denoted by $\mathcal{W}$,

$$\mathcal{W} = \{\varphi \mid \text{there is } \omega \in \Delta^n \text{ s.t. } \varphi = \mathrm{WD}^\omega\};$$

the class of positively weighted division values $\mathcal{W}_+ \subseteq \mathcal{W}$ is given by

$$\mathcal{W}^+ = \left\{\varphi \mid \text{there is } \omega \in \Delta^n_+ \text{ s.t. } \varphi = \mathrm{WD}^\omega\right\}.$$

Note that the constants ω_i, $i \in N$, in the definitions of the weighted division values are exogenously given, *i.e.*, they do not depend on the game v under consideration. The **equal division value (ED-value)** is the positively weighted division value given by

$$\mathrm{ED}_i(v) = \frac{v(N)}{n} \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

The **equal surplus division value (ESD-value)** is the value value given by

$$\mathrm{ESD}_i(v) = v(i) + \frac{1}{n} \cdot \left(v(N) - \sum_{j \in N} v(j)\right) \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

The **Shapley value (Sh-value)** (Shapley, 1953) is given by

$$\mathrm{Sh}_i(v) = \sum_{S \subseteq N : S \ni i} \frac{(n-s)! \cdot (s-1)!}{n!} \cdot (v(S) - v(S \setminus i)) \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

Later on, we will use the following standard axioms for values.

Efficiency. For all $v \in \mathbb{V}$, $\sum_{i \in N} \varphi_i(v) = v(N)$.

Equal treatment axiom. For all $v \in \mathbb{V}$ and $i, j \subseteq N$ such that i and j are substitutes in v, $\varphi_i(v) = \varphi_j(v)$.

Null player axiom. For all $v \in \mathbb{V}$ and $i \in \mathbb{V}$ such that i is null in v, $\varphi_i(v) = 0$.

Nullifying player axiom. For all $v \in \mathbb{V}$ and $i \in N$ such that i is nullifying in v, $\varphi_i(v) = 0$.

Additivity. For all $v, w \in \mathbb{V}$, $\varphi(v + w) = \varphi(v) + \varphi(w)$.

Linearity. For all $v, w \in \mathbb{V}$ and all $c \in \mathbb{R}$, $\varphi(c \cdot v + w) = c \cdot \varphi(v) + \varphi(w)$.

Self-duality. For all $v \in \mathbb{V}$, $\varphi(v) = \varphi(v^D)$.

The Shapley value can be characterized by *efficiency*, *additivity*, *the null player axiom* and *the equal treatment axiom*. Replacing *the null player axiom* by *the nullifying player axiom* yields a characterization of the ED-value (see Theorem 3.1 in van den Brink, 2007).

3. Addition invariance on bi-partitions

The use of bi-partitions of N has been suggested by von Neumann and Morgenstern (1953). Suppose that in a game $v \in \mathbb{V}$ the grand coalition N splits into two coalitions S and $N \setminus S$ that bargain on the surplus $v(N) - v(S) - v(N \setminus S)$ they can create by cooperating. In a sense, the worths $v(S)$ and $v(N \setminus S)$ are the bargaining powers of these two bargaining coalitions. The axiom of *addition invariance on bi-partitions* indicates that if the worths of S and $N \setminus S$ vary by the same amount, then this change should not affect the resulting payoff vector. For $v \in \mathbb{V}$, $\emptyset \subsetneq S \subsetneq N$, and $c \in \mathbb{R}$, the game $v_{S,c} \in \mathbb{V}$ induced by v, S and c is given by

$$v_{S,c}(T) := \begin{cases} v(T) + c, & T \in \{S, N \setminus S\}, \\ v(T), & T \in 2^N \setminus \{S, N \setminus S\} \end{cases} \qquad \text{for all } T \subseteq N. \tag{2}$$

Addition invariance on bi-partitions. For all $v \in \mathbb{V}$, $\emptyset \subsetneq S \subsetneq N$, and $c \in \mathbb{R}$, $\varphi(v) = \varphi(v_{S,c})$.

The next result highlights that *addition invariance on bi-partitions* is equivalent to *self-duality* for additive values.

Lemma 1. *(a) If a value φ satisfies addition invariance on bi-partitions, then φ satisfies self-duality.*

(b) If a value φ satisfies self-duality and additivity, then φ satisfies addition invariance on bi-partitions.

Proof. *(a)*: Let φ be any value that satisfies *addition invariance on bi-partitions*. Let $v \in \mathbb{V}$ and v^D its dual, and define the game $w \in \mathbb{V}$ by $w = (v + v^D)/2$. Now let $i \in N$ and any ordering $(S^1, \ldots, S^{2^{n-1}-1})$ of all coalitions containing player i except N. For all $p \in \{1, \ldots, 2^{n-1} - 1\}$, construct recursively the game v^p by $v^p = (v^{p-1})_{S^p, c^p}$, where $v^0 = v$ and

$$c^p = \frac{v^{p-1}(N) - v^{p-1}(S^p) - v^{p-1}(N \setminus S^p)}{2}.$$

At each step p, we have $v^p(T) = v^{p-1}(T)$ for $T \neq S^p$ or $T \neq N \setminus S^p$,

$$v^p(S^p) = \frac{v(S^p) + v^D(S^p)}{2}, \quad \text{and} \quad v^p(N \setminus S^p) = \frac{v(N \setminus S^p) + v^D(N \setminus S^p)}{2}.$$

As a consequence, we obtain $v^{2^{n-1}-1} = w$. Successive applications of *addition invariance on bi-partitions* yield $\varphi(v) = \varphi(w)$. Considering v^D instead of v, *i.e.*, $v^0 = v^D$, and proceeding in the same fashion, we get $\varphi(v^D) = \varphi(w)$. Therefore, $\varphi(v) = \varphi(v^D)$, as desired.

(b): Let φ be any value that satisfies *self-duality* and *additivity*. Let $v \in \mathbb{V}$, $\emptyset \subsetneq S \subsetneq N$, and $c \in \mathbb{R}$, and the game $v_{S,c}$ induced by v, S and c. Then, $v - v_{S,c} = c \cdot (e_S + e_{N\setminus S})$. In addition, for all $T \subseteq N$, we have $c \cdot e_S^D(T) = -c$ if $T = N \setminus S$ and $c \cdot e_S^D(T) = 0$ if $T \neq N \setminus S$. Therefore, $c \cdot e_{N\setminus S} = -(c \cdot e_S)^D = -c \cdot e_S^D$, and we get $(v - v_{S,c}) = c \cdot (e_S - e_S^D)$. By *additivity* and *self-duality*, we obtain for all $i \in N$, $0 = \varphi_i(c \cdot (e_S - e_S^D))$ and so $0 = \varphi_i(c \cdot (e_S - e_S^D)) = \varphi_i(v - v_{S,c})$. Applying *additivity* once more, we obtain $\varphi_i(v) = \varphi_i(v_{S,c})$ for all $i \in N$, as desired. □

Lemma 1 (b) implies that the Shapley value as well as any weighted division value satisfy *addition invariance on bi-partitions*.

Remark 1. To see why the converse of Lemma 1 (a) fails, let φ be the non-additive value given by

$$\varphi_i(v) = (v(N) - v(N \setminus i) - v(i))^2 \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

This value satisfies *self-duality* but not *addition invariance on bi-partitions*.

By dropping *the equal treatment axiom* and *additivity* from Theorem 3.1 in van den Brink (2007), and adding *addition invariance on bi-partitions* and *linearity*, we provide a characterization of the class of weighted division values.

Theorem 1. *A value φ satisfies efficiency, linearity, the nullifying player axiom, and addition invariance on bi-partitions if and only if $\varphi \in \mathcal{W}$.*

Proof. It is clear that all values $\varphi \in \mathcal{W}$ satisfies *linearity*, *efficiency*, *addition invariance on bi-partitions*, and *the nullifying player axiom*. Reciprocally, let the value φ satisfy *linearity*, *efficiency*, *addition invariance on bi-partitions*, and *the nullifying player axiom*. Let $S \subsetneq N$, $S \neq \emptyset$, and $c \in \mathbb{R}$. Note that $v_{S,c} = v + c \cdot (e_S + e_{N\setminus S})$. By *linearity* and *addition invariance on bi-partitions*, $\varphi(e_S) = -\varphi(e_{N\setminus S})$. Next, let $i \in N$ and $S \ni i$, $S \neq N$. Since i is nullifying in $e_{N\setminus S}$, by *the nullifying player axiom*, we get $\varphi_i(e_{N\setminus S}) = 0$. By *addition invariance on bi-partitions*, $\varphi_i(e_S) = -\varphi_i(e_{N\setminus S}) = 0$. Thus, $\varphi_i(e_S) = 0$ for all $S \neq N$ and all $i \in N$. Since $\{S, N \setminus S\}_{S \ni i, S \neq N} = \{S\}_{\emptyset \subsetneq S \subsetneq N}$, *linearity* and (1) imply

$$\varphi_i(v) = v(N) \cdot \varphi_i(e_N) \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

Set $\omega_i = \varphi_i(e_N)$, $i \in N$, and, by *efficiency*, conclude that $\omega \in \Delta^N$ and $\varphi = \mathrm{WD}^\omega$, *i.e.*, $\varphi \in \mathcal{W}$. □

The necessity to strengthen *additivity* used in Theorem 3.1 in van den Brink (2007) by invoking *linearity* in Theorem 1 and other results is explained in the conclusion of the article. From Lemma 1 and Theorem 1, we get the following corollary.

Corollary 1. *A value φ satisfies efficiency, linearity, the nullifying player axiom, and self-duality if and only if $\varphi \in \mathcal{W}$.*

Most of the characterizations of the ED-value in the literature use *efficiency*, *the equal treatment axiom*, or *additivity*. The following axiom enables a characterization of the ED-value without any of these axioms.

Weak covariance. For all $v \in \mathbb{V}$ and $i \in N$, and all $a, c \in \mathbb{R}$, $\varphi_i(a \cdot v + \mathbf{c}) = a \cdot \varphi_i(v) + c$.

Weak covariance is a weaker version of the classical axiom of *covariance*[3] since in the latter the added additive game is not required to be symmetric. So, any value satisfying *covariance* also satisfies *weak covariance*, while the converse is obviously not true. *Weak covariance* is also imposed by Béal et al. (2015a,b) and van den Brink et al. (2012).

Remark 2. Like any additive value, any value φ satisfying *weak covariance* is an odd function, *i.e.*, $\varphi(-v) = -\varphi(v)$ for all $v \in \mathbb{V}$.

We show that replacing *linearity* and *efficiency* in Theorem 1 by *weak covariance* singles out the ED-value within the class of all weighted values.

Theorem 2. *A value φ satisfies the nullifying player axiom, weak covariance, and addition invariance on bi-partitions if and only if it is the ED-value.*

Proof. One easily checks that the ED-value satisfies *the nullifying player axiom*, *weak covariance* and *addition invariance on bi-partitions*. To prove the uniqueness part, let φ be any value that satisfies *addition invariance on bi-partitions*, *the nullifying player axiom*, and *weak covariance*. Let $v \in \mathbb{V}$ and define the game $v^0 := v + (-v(N)/n) \cdot \sum_{j \in N} u_j$. Note that v^0 is the sum of v and the symmetric additive game induced by $(-v(N)/n)$, and that $v^0(N) = 0$. Now let $i \in N$, and any ordering $(S^1, \ldots, S^{2^{n-1}-1})$ of all coalitions containing i except N. For all $p \in \{1, \ldots, 2^{n-1} - 1\}$ construct recursively the game v^p as $v^p = (v^{p-1})_{S^p, -v(S^p)}$. As a result, the game $v^{2^{n-1}-1}$ is such that $v^{2^{n-1}-1}(S) = 0$ for all coalitions S containing player i. This means that i is a nullifying player in this game and so, by *the nullifying player axiom*, we have $\varphi_i(v^{2^{n-1}-1}) = 0$. By successive applications of *addition invariance on bi-partitions* and *weak covariance*, we get

$$0 = \varphi_i(v^{2^{n-1}-1}) = \varphi_i(v^0) = \varphi_i(v) - \frac{v(N)}{n},$$

i.e., $\varphi_i(v) = \mathrm{ED}_i(v)$. Because v and i were chosen arbitrarily the proof is complete. □

From Lemma 1 (b) and Theorem 2, we obtain the following corollary, for which the logical independence of the axioms is preserved as shown in the appendix.

Corollary 2. *A value φ satisfies the nullifying player axiom, weak covariance, additivity, and self-duality if and only if it is the ED-value.*

4. Null, nullified, and non-negative players

This section invokes three extra axioms, which rest on the notions of the null player, the nullifying player, and on a variant of these types of players. The first of these axioms is introduced by Casajus and Huettner (2013) and requires that if the grand coalition enjoys a non-negative worth, then a null player should not be attributed a negative payoff.

[3] Covariance is also known as *transferable-utility invariance* in Hart and Mas-Colell (1989), *covariance under strategic equivalence* in Peleg and Sudhölter (2003), *zero-independence* in Hokari (2005), and *invariance* in van den Brink (2007), among other names.

Null player in a productive environment axiom. For all $v \in \mathbb{V}$ and $i \in N$ such that $v(N) \geq 0$ and i is a null player in v, $\varphi_i(v) \geq 0$.

Casajus and Huettner (2013) employ *the null player in a productive environment axiom* in order to characterize mixtures between the Shapley value and the ED-value. Dropping *addition invariance on bi-partitions* from Theorem 1 and adding *the null player in a productive environment axiom* selects the positively weighted division values among the set of all weighted division values.

Theorem 3. *A value φ satisfies efficiency, linearity, the nullifying player axiom, and the null player in a productive environment axiom if and only if $\varphi \in \mathcal{W}^+$.*

Proof. It is clear that all values $\varphi \in \mathcal{W}^+$ satisfies *efficiency*, *linearity*, *the null player in a productive environment axiom*, and *the nullifying player axiom*. Reciprocally, let the value φ satisfy *efficiency*, *linearity*, *the null player in a productive environment axiom*, and *the nullifying player axiom*. Firstly, *the nullifying player axiom* implies that $\varphi_i(e_S) = 0$ for all $S \not\ni i$, $S \neq N$. Secondly, we show that $\varphi_i(e_S) = 0$ for all $S \ni i$, $S \neq N$. Let $i \in N$ and $S \ni i$ with $1 < s < n$. Let $w_S^i \in \mathbb{V}$ be given by $w_S^i = e_S + e_{S\setminus i}$. Player i is null in w_S^i and $w_S^i(N) = 0$ since $s < n$. By *linearity* and *the null player in a productive environment axiom*, we get $\varphi_i(e_S) \geq -\varphi_i(e_{S\setminus i})$. Taking $-w_S^i$ instead of w_S^i, we have $-\varphi_i(e_S) \geq \varphi_i(e_{S\setminus i})$. Thus, $\varphi_i(e_S) = -\varphi_i(e_{S\setminus i})$. In each standard game $e_{S\setminus i}$, player $i \in S \subseteq N$ is nullifying, so that *the nullifying player axiom* yields $\varphi_i(e_{S\setminus i}) = 0$. It follows that $\varphi_i(e_S) = -\varphi_i(e_{S\setminus i}) = 0$ for all $S \ni i$ such that $1 < s < n$. Moreover, since all $j \in N \setminus i$ are nullifying in e_i, we also get $\varphi_j(e_i) = 0$. Thus, applying *efficiency* in e_i implies that $\varphi_i(e_i) = 0$ as well. As a consequence, $\varphi_i(e_S) = 0$ for all $S \neq N$ and all $i \in N$ as claimed, which implies that (1) can be rewritten as

$$\varphi_i(v) = v(N) \cdot \varphi_i(e_N) \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

By *the null player in a productive environment axiom*, we obtain $\varphi_i(e_N + e_{N\setminus i}) \geq 0$. By *linearity*, $\varphi_i(e_N) \geq -\varphi_i(e_{N\setminus i})$, and by *the nullifying player axiom*, $\varphi_i(e_{N\setminus i}) = 0$ so that $\varphi_i(e_N) \geq 0$. Set $\omega_i = \varphi_i(e_N) \geq 0$ for all $i \in N$ and, by *efficiency*, conclude that $\varphi \in \mathcal{W}^+$. □

The second axiom defined in this section is new. It aims at emphasizing that a player is responsible for the worths of the coalitions he belongs to. The axiom stipulates that if the worths of all coalitions to which a given player belongs are non-negative, then this player should get at least a zero payoff.

Non-negative player axiom. For all $v \in \mathbb{V}$ and $i \in N$ such that i is a non-negative player in v, $\varphi_i(v) \geq 0$.

Similarly to *the nullifying player axiom*, the *non-negative player axiom* is based on the worth of a player's coalitions instead of the player's marginal contribution to coalitions. *The nullifying player axiom* specifies the nullifying player's payoff. The requirement in *the non-negative player axiom* is somehow weaker in the sense that the non-negative player's payoff is not completely specified, allowing two non-negative players to obtain different payoffs. As such, *the non-negative player axiom* is related to *the nullifying player axiom* in a similar way as *the null player axiom* is related to *the null player in a productive environment axiom*. Replacing *the null player in a productive environment axiom* and *the nullifying player axiom* in Theorem 3 by *the non-negative player axiom* gives an alternative characterization of the positively weighted division values.

Theorem 4. *A value φ satisfies efficiency, linearity, and the non-negative player axiom if and only if $\varphi \in \mathcal{W}^+$.*

Proof. In a game $v \in \mathbb{V}$, a player $i \in N$ is non-negative only if $v(N) \geq 0$. Thus, any value $\varphi \in \mathcal{W}^+$ satisfies *the non-negative player axiom*. For the uniqueness part, consider any value satisfying the three axioms. For $S \neq N$, all players are non-negative in e_S. By *the non-negative player axiom*, $\varphi_i(e_S) \geq 0$ for all $i \in N$. By *efficiency*, it must be that $\varphi_i(e_S) = 0$ for all $i \in N$. In e_N, *the non-negative player axiom* also implies $\varphi_i(e_N) \geq 0$ for all $i \in N$. Set $\omega_i = \varphi_i(e_N) \geq 0$ for all $i \in N$. Conclude by *efficiency*, *linearity*, and (1) that $\varphi \in \mathcal{W}^+$. $\square$

The third axiom defined in this section incorporates a solidarity principle. *Nullified solidarity* (Béal et al., 2014) compares a game before and after a specified player becomes null in the sense that he now has a null contribution to all coalition he belongs to. The axiom simply requires uniformity in the direction of the payoff variation for all players in the situations where the considered player loses from being nullified. As such, *nullified solidarity* is silent, *a priori*, on what happens if this player increases his payoff after being nullified. Formally, for a game $v \in \mathbb{V}$ and a player $i \in N$, the associated game in which i is nullified, denoted by $v^{\mathbf{N}i} \in \mathbb{V}$, is given by

$$v^{\mathbf{N}i}(S) = v(S \setminus i) \qquad \text{for all } S \subseteq N. \tag{3}$$

Nullified solidarity. For all $v \in \mathbb{V}$ and $i, j \in N$, $\varphi_i(v) \geq \varphi_i(v^{\mathbf{N}i})$ implies $\varphi_j(v) \geq \varphi_j(v^{\mathbf{N}i})$.

Nullified solidarity has the same flavor as the axiom of *population solidarity* proposed by Chun and Park (2012), which requires that if some players leave a game, then the remaining players should be affected in the same direction. When a player is nullified, he does not exactly leave the game, but his presence or absence in a coalition has no impact of the achieved worths. In Chun and Park (2012), *population solidarity* belongs to the set of axioms characterizing the ESD-value on the class of games with variable player sets. Beyond the aforementioned similarities, the two axioms are not logically related to each other. The ESD-value satisfies *population solidarity* but not *nullified solidarity*. The value which assigns to a player his/her stand alone worth times the number of players in the game satisfies *nullified solidarity* but not *population solidarity*.

The next result shows that *nullified solidarity* can be used as a substitute to *the non-negative player axiom* in Theorem 4 in order to provide another characterization of the positively weighted division values.

Theorem 5. *The value φ satisfies efficiency, linearity, and nullified solidarity if and only if $\varphi \in \mathcal{W}^+$.*

Proof. Any value $\varphi \in \mathcal{W}^+$ satisfies the three axioms. For the uniqueness part, let φ be any value that satisfies the three axioms. For all $S \subseteq N$ and $i \in S$, $(e_S)^{\mathbf{N}i} = \mathbf{0}$. By *linearity*, $\varphi_j((e_S)^{\mathbf{N}i}) = \varphi_j(\mathbf{0}) = 0$ for all $j \in N$. Next, we show that $\varphi_i(e_S) \geq 0$ for all $S \subseteq N$ and all $i \in S$. Assume by contradiction that there are $S \subseteq N$ and $i \in S$ such that $\varphi_i(e_S) < 0$. Consider the game $-e_S$. By *linearity*, we get $\varphi_i(-e_S) = -\varphi_i(e_S) > 0$, and of course $(-e_S)^{\mathbf{N}i} = (e_S)^{\mathbf{N}i} = \mathbf{0}$. Thus, $\varphi_i(-e_S) > \varphi_i((-e_S)^{\mathbf{N}i})$. By *nullified solidarity*, this implies that $\varphi_j(-e_S) \geq 0$ for all $j \in N \setminus i$. Summing on all $j \in N$, we obtain $\sum_{j \in N} \varphi_j(-e_S) > 0$, or equivalently $\sum_{j \in N} \varphi_j(e_S) < 0$, a contradiction with the fact that φ satisfies *efficiency*. In other words, the inequality $\varphi_i(e_S) \geq 0$ for all $S \subseteq N$ and $i \in S$ is true. Then, by *nullified solidarity*, $\varphi_i(e_S) \geq \varphi_i((e_S)^{\mathbf{N}i}) = 0$ implies $\varphi_j(e_S) \geq \varphi_j((e_S)^{\mathbf{N}i}) = 0$ for all $j \in N$. It remains to distinguish two cases. Firstly, suppose that $S \neq N$. By *efficiency* and $e_S(N) = 0$, we get $\varphi_j(e_S) = 0$ for all $j \in N$. Secondly, suppose that $S = N$. Set $\omega_j = \varphi_j(e_N) \geq 0$ for all $j \in N$. By *efficiency*, $\sum_{j \in N} \omega_j = 1$. By *linearity* and (1), the proof is complete. $\square$

Replacing *linearity* in Theorem 5 by *weak covariance* singles out the ED-value from the set of positively weighted division values. This result shows that the symmetric treatment imposed by *weak covariance* in the added additive game clearly plays a decisive role, even if *linearity* is not required. This result echoes Theorem 2 in Béal et al. (2014), in which the ED-value is characterized by *efficiency*, *nullified solidarity*, together with the following two axioms of *null game* and *weak fairness*.

Null game. For all $i \in N$, $\varphi_i(\mathbf{0}) = 0$.

Weak fairness. For all $v, w \in \mathbb{V}$ and $c \in \mathbb{R}$ such that $w(S \cup i) - w(S) = v(S \cup i) - v(S) + c$ for all $i \in N$ and $S \subseteq N \setminus i$, $\varphi_i(v) - \varphi_i(w) = \varphi_j(v) - \varphi_j(w)$ for all $i, j \in N$.

Null game is a classical axiom. *Weak fairness* states that all players should gain or lose equally, whenever all marginal contributions to coalitions of all players are changed by the same amount. The principle behind *weak fairness* originates from the axiom of *fairness* introduced by van den Brink (2001). The latter axiom requires that if to a game another game is added in which *two* players are substitutes then their payoffs change by the same amount. As demonstrated in the proof of Theorem 6, the game w is obtained from the game v by adding a symmetric additive game $\mathbf{c}$. Therefore, *Weak fairness* is similar to *fairness* in the sense that to a game we add a (specific) game in which *all* players are substitutes whereas only two substitute players are needed in *fairness*. Any value satisfying *fairness* also satisfies *weak fairness*, while the converse implication does not hold.

Theorem 6. *A value φ satisfies efficiency, nullified solidarity, and weak covariance if and only if it is the ED-value.*

Proof. The ED-value clearly satisfies all axioms. Thus, by Theorem 2 in Béal et al. (2014), it is enough to show that *weak covariance* implies both *null game* and *weak fairness*. The first implication follows from the definition of *weak covariance* by setting $a = c = 0$. For the second implication, let $v, w \in \mathbb{V}$ as described in *weak fairness*. We show that $w = v + \mathbf{c}$, *i.e.* that $w(S) = v(S) + s \cdot c$ for all $S \subseteq N$. We proceed by induction on s. For coalitions of size 1, it is obvious that $w(i) = v(i) + c$. So assume that $w(S) = v(S) + s \cdot c$ is true for all $S \subseteq N$ such that $s < k$ for some $k \in \{2, \ldots, n\}$. Now, take any $S \subseteq N$ such that $s = k$, and let $i \in S$. By definition of v and w, and the induction hypothesis, we can write that

$$w(S) = v(S) + w(S \setminus i) - v(S \setminus i) + c \iff w(S) = v(S) + (s-1) \cdot c + c \iff w(S) = v(S) + s \cdot c.$$

As a consequence, *weak covariance* can be applied to games v and w to obtain, for all $i \in N$, $\varphi_i(w) = \varphi_i(v) + c$. Thus, for all $i, j \in N$, we get $\varphi_i(w) - \varphi_i(v) = \varphi_j(w) - \varphi_j(v)$ as desired, completing the proof. □

Note that the combination of *null game* and *weak fairness* does not imply *weak covariance*. For instance, the value $\varphi = 2 \cdot \mathrm{ED}$ satisfies both *null game* and *weak fairness* but violates *weak covariance*.

5. A comparison with van den Brink (2007)

This section deals with another advantage of some of our results over the characterization of the ED-value proposed in Theorem 3.1 in van den Brink (2007) . The remark below states that dropping *the equal treatment axiom* from Theorem 3.1 in van den Brink (2007) does not ensure that the resulting values are weighted division values, even if *linearity* is imposed instead of *additivity*.

Remark 3. The set of values satisfying *efficiency*, *additivity* or *linearity*, and *the nullifying player axiom* is not contained in $\mathcal{W}$. In order to see this, it suffices to exhibit a value not in $\mathcal{W}$ satisfying *linearity*, *efficiency*, and *the nullifying player axiom*. For all non-empty $S \subsetneq N$, let $i(S) \in S$. Define the value φ as

$$\varphi_i(v) = \text{ED}_i(v) + \sum_{S:i(S)=i} v(S) - \sum_{S:S\setminus i(S)\ni i} \frac{v(S)}{s-1} \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N. \tag{4}$$

Since the family $(i(S))_{\emptyset\subsetneq S\subseteq N}$ does not depend on $v \in \mathbb{V}$, φ satisfies *linearity*. Next, for all non-empty $S \neq N$, $\sum_{i\in N} \varphi_i(e_S) = 0$ and $\sum_{i\in N} \varphi_i(e_N) = 1$. Using (1), conclude that φ satisfies *efficiency*. For all v and all $i \in N$, $\varphi_i(v)$ depends only on the worth of coalitions containing player i, so that φ obviously satisfies *the nullifying player axiom*. However, φ cannot belong to $\mathcal{W}$.

In a sense, the role of *the equal treatment axiom* in Theorem 3.1 in van den Brink (2007) is not only to assign an identical share of the worth of grand coalition but also to neutralize the influence of all smaller coalitions on the distribution of payoffs. Indeed, we can use two of our results to provide characterizations of the ED-value in which dropping *the equal treatment axiom* yields the set of positively weighted division values. To understand this aspect, consider the following statement.

Theorem 7. *A value φ satisfies equal treatment axiom and either*
(a) efficiency, additivity, and the non-negative player axiom,
(b) efficiency, additivity, and nullified solidarity,
if and only if it is the ED-value.

Replacing *additivity* in Theorem 7 by *linearity* still generates two sets of logically independent axioms (see the appendix for more details). Proceeding in this fashion and dropping *the equal treatment axiom* as we did in Theorem 3.1 in van den Brink (2007) to obtain Remark 3, we recover the characterizations of the positively weighted division values provided by Theorem 4 and 5, respectively. Another view on the results in this section is to remark that Theorem 3.1 in van den Brink (2007) and Theorem 7 (a) and (b) only differ with respect to one axiom, *the nullifying player axiom*, *the non-negative player axiom*, and *nullified solidarity*, respectively. In a sense, *the nullifying player axiom* is not strong enough to generate only positively weighted division values without the help of *the equal treatment axiom* as it is the case with *the non-negative player axiom* and *nullified solidarity*.

Proof. (Theorem 7) We shall only prove the uniqueness parts. For part (a), if *additivity* replaces *linearity* and $c \cdot e_S$, $c \in \mathbb{R}$, replaces e_S in the proof of Theorem 4, we still can conclude that $\varphi_i(c \cdot e_S) = 0$ for all $S \neq N$, $i \in N$ and $c \in \mathbb{R}$ since φ remains an odd function. As a consequence, the following representation of any additive value

$$\varphi_i(v) = \sum_{S\subseteq N} \varphi_i(v(S) \cdot e_S) \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N$$

can be rewritten as

$$\varphi_i(v) = \varphi_i(v(N) \cdot e_N) \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

Finally, *efficiency* and *equal treatment* imply $\varphi_i(v(N) \cdot e_N) = v(N)/n$, as desired.

Referring to Theorem 5, the proof part (b) is very much the same as for part (a). □

6. Concluding remarks

We conclude this article with one remark and a recap chart. The reader might wonder whether *linearity* can be weakened by using *additivity* in Theorem 1, 3, 4 and 5 and Corollary 1, especially because this is exactly what is done in Theorem 7 in a different context. This is not possible. The reason is that there exist additive functions which are not linear, and that *linearity* cannot be derived from the combination of *additivity* and the other axioms. As an illustration, let us focus on Theorem 1 in order to show that there are values, outside the set of weighted division values that satisfy *additivity*, *efficiency*, *addition invariance on bi-partitions*, and *the nullifying player axiom*. As suggested in the proof of Theorem 7, replacing *linearity* by *additivity* yields that the value under consideration can be written as

$$\varphi_i(v) = \varphi_i(v(N) \cdot e_N) \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

Now, choose a function $f : \mathbb{R} \longrightarrow \mathbb{R}$, which is additive but not linear (Macho-Stadler et al., 2007, p. 352, also consider such a function). Using f, define the non-linear value φ by

$$\varphi_i(v) = \begin{cases} \mathrm{ED}_i(v) + (-1)^i \cdot f(v(N)), & i \in \{1, 2\}, \\ \mathrm{ED}_i(v), & i \in N \setminus \{1, 2\} \end{cases} \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

Note that f cannot be null everywhere on its domain since otherwise it would be linear. As a consequence, the value φ does not belong to the set of weighted division values even though it satisfies *additivity*, *efficiency*, *addition invariance on bi-partitions*, and *the nullifying player axiom*.

The characterizations contained in this article are summarized in the following table, in which a "+" means that a value satisfies the axiom, in which "−" has the converse meaning, and in which the "⊕" symbols indicate the axioms used in the corresponding characterization. Also, Theorems and Corollaries are abbreviated by letters T and C respectively, followed by their identifying number. T3.1 refers to Theorem 3.1 in van den Brink (2007). Lastly, the ESD-value and Sh-value are added to the table in order to point out which of our axioms they satisfy.

	$\mathcal{W}$		$\mathcal{W}^+$			ED						ESD	Sh
	T1	C1	T3	T5	T4	T3.1	T2	C2	T6	T7a	T7b		
Efficiency	⊕	⊕	⊕	⊕	⊕	⊕	+	+	⊕	⊕	⊕	+	+
Equal treatment	−	−	−	−	−	⊕	+	+	+	⊕	⊕	+	+
Nullifying player	⊕	⊕	⊕	+	+	⊕	⊕	⊕	+	+	+	−	−
Additivity	+	+	+	+	+	⊕	+	⊕	+	⊕	⊕	+	+
Linearity	⊕	⊕	⊕	⊕	⊕	+	+	+	+	+	+	+	+
Self-duality	+	⊕	+	+	+	+	+	⊕	+	+	+	−	+
Addition invariance on bi-partitions	⊕	+	+	+	+	+	⊕	+	+	+	+	−	+
Weak covariance	−	−	−	−	−	+	⊕	⊕	⊕	+	+	+	+
Null player in a productive environment	−	−	⊕	+	+	+	+	+	+	+	+	−	+
Non-negative player	−	−	+	+	⊕	+	+	+	+	⊕	+	−	−
Nullified solidarity	−	−	+	⊕	+	+	+	+	⊕	+	⊕	−	−
Null game	+	+	+	+	+	+	+	+	+	+	+	+	+
Weak fairness	−	−	−	−	−	+	+	+	+	+	+	+	+

AppendixA. Logical independence of the axioms in the characterizations

We focus on non-trivial cases, *i.e.*, if $n > 1$ or $n > 2$. In each of the following proofs, we exhibit a value that satisfies all of the axioms in one of our characterizations except for the one that is named. Details are provided for the toughest cases.

FOR THEOREM 1:

Not *efficiency*: the null value;
Not *linearity*: the value φ defined by

$$\varphi_i(v) = (v(i) - v(N \setminus i)) \cdot v(N) \quad \text{and} \quad \varphi_1(v) = \left(1 - \sum_{i \in N \setminus 1} [v(i) - v(N \setminus i)]\right) \cdot v(N)$$

for all $v \in \mathbb{V}$ and $i \in N \setminus \{1\}$;
Not *the nullifying player axiom*: Sh-value;
Not *addition invariance on bi-partitions*: the value φ defined by (4) in Remark 3.

FOR COROLLARY 1:

Not *efficiency*: the null value;
Not *linearity*: the value φ defined by

$$\varphi_i(v) = (v(i) - v(N \setminus i)) \cdot v(N) \quad \text{and} \quad \varphi_1(v) = \left(1 - \sum_{i \in N \setminus 1} [v(i) - v(N \setminus i)]\right) \cdot v(N)$$

for all $v \in \mathbb{V}$ and $i \in N \setminus \{1\}$;
Not *the nullifying player axiom*: Sh-value;
Not *self-duality*: the value φ defined by (4) in Remark 3.

FOR THEOREM 2:

Not *the nullifying player axiom*: Sh-value;
Not *weak covariance*: any value $\varphi \in \mathcal{W}^+ \setminus \{\text{ED}\}$;
Not *addition invariance on bi-partitions*: the value φ defined by $\varphi_i(v) = v(i)$ for all $v \in \mathbb{V}$ and $i \in N$.

FOR COROLLARY 2:

Not *the nullifying player axiom*: Sh-value;
Not *weak covariance*: any value $\varphi \in \mathcal{W}^+ \setminus \{\text{ED}\}$;
Not *additivity*: note that a game v is additive but not symmetric if there exists a weight vector $(c_1, \ldots, c_n) \in \mathbb{R}^n$ with not all identical coordinates and such that $v = \sum_{i \in N} c_i u_i$. Let A be the class of all games on N that are additive but not symmetric. Define the value φ by

$$\varphi_i(v) = \begin{cases} v(i), & v \in A, \\ \text{ED}_i(v), & v \in \mathbb{V} \setminus A \end{cases} \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

Note that for all $a, c \in \mathbb{R}$, $v \in A$ if and only if $(a \cdot v + \mathbf{c}) \in A$, $a \neq 0$, *i.e.*, the class of all additive but not symmetric games on N is closed under the "$(a \cdot v + \mathbf{c})$-operation", provided that $a \neq 0$. If $a = 0$ then $(a \cdot v + \mathbf{c}) = \mathbf{c}$ but in this case, for all $i \in N$, $\mathrm{ED}_i(\mathbf{c}) = c = \mathbf{c}(i)$. As a consequence, φ satisfies *weak covariance*. For any additive game, observe that $v^D = v$, so that $v \in A$ if and only if $v^D \in A$. In particular, we have $v(i) = v(N) - v(N \setminus i) = v^D(i)$. This implies that φ satisfies *self-duality*. It is also easy to check that φ satisfies *the nullifying player axiom*. Finally, let $v \in A$, *i.e.*, $v = \sum_{j \in N} c_j \cdot u_j$ with $c_i \neq c_j$ for some $i, j \in N$. For any given $c \in \mathbb{R} \setminus \{0\}$, both games $c \cdot e_N$ and $v - c \cdot e_N$ are not additive, and thus not in A. It follows that, for all $i \in N$, $\varphi_i(v - c \cdot e_N) = (v(N) - c)/n$ and $\varphi_i(c \cdot e_N) = c/n$. Therefore, $\varphi_i(v - c \cdot e_N) + \varphi_i(c \cdot e_N) = v(N)/n$ for all $i \in N$, *i.e.*, all players get the same payoff in the sum of the two games. But $\varphi_i(v - c \cdot e_N + c \cdot e_N) = \varphi_i(v) = v(i) = c_i$ for all $i \in N$ which implies that not all players get the same payoff in game $v - c \cdot e_N + c \cdot e_N$, proving that φ does not satisfy *additivity*;

Not *addition invariance on bi-partitions*: the value φ defined by $\varphi_i(v) = v(i)$ for all $v \in \mathbb{V}$ and $i \in N$.

For Theorem 3:

Not *efficiency*: the value φ defined by $\varphi_i(v) = v(i)$ for all $v \in \mathbb{V}$ and all $i \in N$;

Not *linearity*: the value φ defined by

$$\varphi_i(v) = \begin{cases} \dfrac{v(i)^2}{\sum_{j \in N} v(j)^2} \cdot v(N) & \text{if } \sum_{j \in N} v(j)^2 \neq 0, \\ \mathrm{ED}_i(v) & \text{if } \sum_{j \in N} v(j)^2 = 0 \end{cases} \tag{A.1}$$

for all $v \in \mathbb{V}$ and $i \in N$;

Not *the nullifying player axiom*: Sh-value;

Not *the null player in a productive environment axiom*: any value $\varphi \in \mathcal{W} \setminus \mathcal{W}^+$.

For Theorem 4:

Not *efficiency*: the null value;

Not *linearity*: the value given by (A.1);

Not *the non-negative player axiom*: Sh-value.

For Theorem 5:

Not *efficiency*: the null value;

Not *linearity*: let $\omega \in \mathbb{R}^N$ be such that $\sum_{i \in N} \omega_i = 0$ and $\omega_i \neq 0$ for some $i \in N$. Construct the value φ defined by $\varphi_i(v) = \mathrm{ED}_i(v) + \omega_i$ for all $v \in \mathbb{V}$ and $i \in N$;

Not *nullified solidarity*: Sh-value.

For Theorem 6:

Not *efficiency*: any value $\varphi \in \mathcal{W}^+ \setminus \{\mathrm{ED}\}$;

Not *weak covariance*: for some $i \in N$, the value $\varphi^{(i)}$ defined by $\varphi^{(i)}_j(v) = v(i)$ for all $v \in \mathbb{V}$ and all $j \in N$;

Not *nullified solidarity*: Sh-value.

For Theorem 7 *(a):*

Not *the equal treatment axiom*: any value $\varphi \in \mathcal{W}^+ \setminus \{\mathrm{ED}\}$;
Not *efficiency*: the null value;
Not *additivity*: value given by (A.1);
Not *the non-negative player axiom*: Sh-value.

For Theorem 7 *(b):*

Not *the equal treatment axiom*: any value $\varphi \in \mathcal{W}^+ \setminus \{\mathrm{ED}\}$;
Not *efficiency*: the null value;
Not *additivity*: Suppose that $n \geq 3$. Let $w \in \mathbb{V}$ be such that no two distinct players are substitutes,

$$w(N) > 0, \quad \text{and} \quad w(N \setminus i) = 0 \quad \text{for all } i \in N. \tag{A.2}$$

Let $\omega \in \mathbb{R}^n_+$ such that $\sum_{i\in N} \omega_i = 1$ and $\omega_i \neq \omega_j$ for some $i, j \in N$. Define the value φ by $\varphi_i(w) = \mathrm{WD}^\omega_i(w)$ and $\varphi_i(v) = \mathrm{ED}_i(v)$ if $v \in \mathbb{V} \setminus \{w\}$. Since w does not contain any pair of substitute players, φ satisfies *the equal treatment axiom*. It is also obvious that φ satisfies *efficiency*. Regarding *nullified solidarity*, let $v \in \mathbb{V}\setminus\{w\}$. Since condition (A.2) implies that w does not contain any null player, we have $v^{\mathbf{N}i} \neq w$ for all $i \in N$, so that *nullified solidarity* is satisfied when the considered game is $v \in \mathbb{V}\setminus\{w\}$. Now, let us test *nullified solidarity* starting with game w. By (A.2), we have $w^{\mathbf{N}i}(N) = w(N \setminus i) = 0$ for all $i \in N$. Therefore,

$$\varphi_i(w) = \mathrm{WD}^\omega_i(w) \geq 0 = \mathrm{ED}_i(w^{\mathbf{N}i}) = \varphi_i(w^{\mathbf{N}i}),$$

but also

$$\varphi_j(w) = \mathrm{WD}^\omega_j(w) \geq 0 = \mathrm{ED}_j(w^{\mathbf{N}j}) = \varphi_j(w^{\mathbf{N}i}),$$

for all $j \in N\setminus i$, which shows that φ satisfies *nullified solidarity*. Finally, by considering two games v^1 and v^2 such that $v^1 \neq \mathbf{0}$, $v^2 \neq \mathbf{0}$ and $v^1 + v^2 = w$, it is easy to see that φ does not satisfies *additivity*;
Not *nullified solidarity*: Sh-value.

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