Solidarity and fair taxation in TU games

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Abstract

We consider an analytic formulation of the class of efficient, linear, and symmetric values for TU games that, in contrast to previous approaches, which rely on the standard basis, rests on the linear representation of TU games by unanimity games. Unlike most of the other formulae for this class, our formula allows for an economic interpretation in terms of taxing the Shapley payoffs of unanimity games. We identify those parameters for which the values behave economically sound, i.e., for which the values satisfy desirability and positivity. Put differently, we indicate requirements on fair taxation in TU games by which solidarity among players is expressed.

Keywords: Shapley value, solidarity, taxation, desirability, positivity, acceptability

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1. Introduction

The Shapley value [Shapley, 1953] probably is the most eminent point-solution concept for TU games. Its standard characterization involves four axioms: efficiency, additivity/linearity, symmetry, and the null player axiom. In a sense, it is mainly the latter property that prevents the Shapley value to allow for solidarity among the players. Irrespective of the productivity within the whole society, an unproductive player obtains a zero payoff. Moreover, together with additivity, the null player property already entails marginality [Young, 1985], i.e., the players’ payoffs depend only on their own productivities measured by marginal contributions, respectively.

So, if one wishes values to allow for solidarity considerations, one has to drop the null player axiom from the list of required properties. But then we were down to the class of values obeying efficiency, linearity, and symmetry. Obviously, this large class contains a lot of values that deviate from the Shapley value not only by economically sound solidarity considerations. For example, the equal surplus division value [Driessen and Funaki, 1991]...
and the consensus value (Ju et al., 2007) inhabit this class, but fail positivity (Kalai and Samet, 1987). That is, these values may assign negative payoffs in monotonic games, i.e., in games where no player ever is destructive in terms of his marginal contributions. We feel that this does not fit well our intuitions on solidarity. Moreover, values in this class may not meet desirability (Maschler and Peleg, 1966), i.e., a player who is more productive than another one may end up with a lower payoff. Again, this would overstretch our sense of solidarity.

Formulae for the class of efficient, linear, and symmetric values (ELS values) have been proposed by Ruiz et al. (1998), Driessen and Radzik (2003), Chameni-Nembua and Andjiga (2008), and Hernandez-Lamoneda et al. (2008). Recently, Chameni-Nembua (2012) and Malawski (2013) come up with a more interpretational one. In essence, the players’ marginal contributions within a coalition are taxed at a rate depending on its size, while the tax revenue is distributed evenly among the other players in the coalition under consideration.

We suggest and explore an alternative formula for this class, already indicated by Radzik and Driessen (2009, p. 5), which also is interpretable in terms of taxation. The main idea of our approach is to tax and redistribute the Shapley payoffs of unanimity games. First, the Shapley payoffs are taxed at a certain rate, which depends on the cardinality of the set of productive players in such a game. And second, the overall tax revenue is distributed evenly among all players. Linearity extends these payoffs to general TU games. Radzik and Driessen (2013) provide conditions on the parameters of the formula due to Driessen and Radzik (2003) such that the resulting value satisfies one or another of the desirable properties above: desirability and positivity combined with desirability.

In this paper, we attempt analogous conditions on the parameters of our formula. First, we prepare our main results by showing the relation between the parameters of our formula and the parameters of the Driessen-Radzik formula (Propositions 3 and 4). Then, we identify those parameters that entail desirability (Theorem 5). Since there seems to be no nice way to describe the parameters that yield positivity, we identify those parameters that entail positivity for null players both for the Driessen-Radzik formula (Theorem 7) and for our formula (Theorem 8). Combining the afore-mentioned results, we obtain requirements on the parameters that imply both desirability and positivity (Theorem 11). Finally, we identify the parameters that guarantee the acceptability properties suggested by Joosten et al. (1994) and by Radzik and Driessen (2013) (Propositions 16 and 18).

This paper is organized as follows: In the second section, we introduce basic definitions and notation. The third section surveys formulae for ELS values and introduces a new parametrization for this class. In section four, we provide conditions on the parameters of our formulae such that one or another of the desirable properties mentioned above are satisfied. The appendix contains the lengthier proofs.

2. Basic definitions and notation

A (TU) game is a pair \((N, v)\) consisting of a non-empty and finite set of players \(N\) and a coalition function \(v \in \mathbb{V} (N) := \{ f : 2^N \rightarrow \mathbb{R} \mid f(\emptyset) = 0 \}\). Since we work within a fixed player set, we frequently drop the player set as an argument. In particular, we address \(v \in \mathbb{V}\) as a game. Subsets of \(N\) are called coalitions; \(v(S)\) is called the worth of coalition
S. For \( v, w \in \mathbb{V} \) and \( \lambda \in \mathbb{R} \), the coalition functions \( v + w \in \mathbb{V} \) and \( \lambda \cdot v \in \mathbb{V} \) are given by 
\[
(v + w)(S) = v(S) + w(S) \quad \text{and} \quad (\lambda \cdot v)(S) = \lambda \cdot v(S)
\]
for all \( S \subseteq N \). For \( T \subseteq N \), \( T \neq \emptyset \), the game \( u_T \in \mathbb{V} \), \( u_T(S) = 1 \) if \( T \subseteq S \) and \( u_T(S) = 0 \) for \( T \not\subseteq S \), is called a \textit{unanimity game}. For \( T \subseteq N \), \( T \neq \emptyset \), the game \( e_T \in \mathbb{V} \), \( e_T(S) = 1 \) if \( T = S \) and \( e_T(S) = 0 \) for \( T \neq S \), is called a \textit{standard game}. A game \( v \) is called \textit{monotonic} if \( v(S) \geq v(T) \) for all \( S, T \subseteq N \) such that \( T \subseteq S \). Any \( v \in \mathbb{V} \) can be uniquely represented by unanimity games,
\[
v = \sum_{T \subseteq N: T \neq \emptyset} \lambda_T(v) \cdot u_T,
\]
where the Harsanyi dividends, \( \lambda_T(v), T \subseteq N, T \neq \emptyset \) (Harsanyi, 1959) are given implicitly by
\[
v(S) = \sum_{T \subseteq S: T \neq \emptyset} \lambda_T(v) \quad \text{for all} \; S \subseteq N, \; S \neq \emptyset.
\]

For \( v \in \mathbb{V}, i \in N, \) and \( S \subseteq N \setminus \{i\} \), the \textit{marginal contribution} of \( i \) to \( S \) in \( v \) is given by \( MC_i^v(S) := v(S \cup \{i\}) - v(S) \). Player \( i \in N \) is called a \textit{null player} in \( v \in \mathbb{V} \) if \( MC_i^v(S) = 0 \) for all \( S \subseteq N \setminus \{i\} \); players \( i, j \in N \) are called \textit{symmetric} in \( v \in \mathbb{V} \) if \( MC_i^v(S) = MC_j^v(S) \) for all \( S \subseteq N \setminus \{i, j\} \).

A \textit{value} on \( N \) is an operator \( \varphi \) that assigns a payoff vector \( \varphi(v) \in \mathbb{R}^N \) to any \( v \in \mathbb{V} \). For \( S \subseteq N \), we denote \( \sum_{i \in S} \varphi_i(v) \) by \( \varphi_S(v) \). The \textbf{Shapley value} (Shapley, 1953), \( \text{Sh} \), given by
\[
\text{Sh}_i(v) := \sum_{T \subseteq N: i \in T} \frac{\lambda_T(v)}{|T|} \quad \text{for all} \; i \in N, \; v \in \mathbb{V}
\]
is the unique value on \( N \) that satisfies the axioms \( \textbf{E, A (or L), S, and N} \) below.

\textbf{Efficiency, E.} For all \( v \in \mathbb{V}, \varphi_N(v) = v(N) \).

\textbf{Additivity, A.} For all \( v, w \in \mathbb{V}, \varphi(v + w) = \varphi(v) + \varphi(w) \).

\textbf{Null player, N.} For all \( v \in \mathbb{V} \) and all \( i \in N \), who are null players in \( v, \varphi_i(v) = 0 \).

We further refer to the following standard axioms.

\textbf{Linearity, L.} For all \( v, w \in \mathbb{V} \) and \( \lambda \in \mathbb{R}, \varphi(v + w) = \varphi(v) + \varphi(w) \) and \( \varphi(\lambda \cdot v) = \lambda \cdot \varphi(v) \).

\textbf{Symmetry, S.} For all \( v \in \mathbb{V}, i \in N, \) and all bijections \( \pi : N \rightarrow N, \varphi_{\pi(i)}(v \circ \pi^{-1}) = \varphi_i(v) \).

\textbf{Continuity, C.} The mapping \( \varphi : \mathbb{V} \rightarrow \mathbb{R}^N \) is continuous.

Moreover, we refer to the following values, which also obey \( \textbf{E, L, and S} \). The \textbf{equal division value}, \( \text{ED} \), is given by
\[
\text{ED}_i(v) := \frac{v(N)}{|N|} \quad \text{for all} \; i \in N, \; v \in \mathbb{V}.
\]

The \textbf{egalitarian Shapley values} (Joosten, 1996), \( \text{Sh}^\alpha, \alpha \in [0, 1] \), are given by \( \text{Sh}^\alpha = \alpha \cdot \text{Sh} + (1 - \alpha) \cdot \text{ED} \). The \textbf{equal surplus division value} (Driessen and Funaki, 1991), \( \text{ES} \), is given by
\[
\text{ES}_i(v) := v(\{i\}) + \frac{v(N) - \sum_{j \in N} v(\{j\})}{|N|} \quad \text{for all} \; i \in N, \; v \in \mathbb{V}.
\]
The **solidarity value** [Nowak and Radzik (1994)](#), \( \text{So}_i(v) \), is given by
\[
\text{So}_i(v) := \sum_{S \subseteq N, i \in S} \frac{1}{|N|} \sum_{j \in S} \frac{v(S) - v(S \setminus \{j\})}{|S|} \quad \text{for all } i \in N, \ v \in \mathbb{V}.
\]

The **consensus value** [Ju et al. (2007)](#), \( \text{Con} \), is given by \( \text{Con} = \frac{1}{2} \cdot \text{Sh} + \frac{1}{2} \cdot \text{ES} \). The **least-square pre-nucleolus** [Ruiz et al. (1996)](#), \( \text{LSPN}_i(v) \), is given by
\[
\text{LSPN}_i(v) := \text{Ba}_i(v) + \frac{v(N) - \sum_{j \in N} \text{Ba}_j(v)}{|N|} \quad \text{for all } i \in N, \ v \in \mathbb{V},
\]
where \( \text{Ba} \) stands for the Banzhaf value [Banzhaf (1965), Owen (1975)].

\[
\text{Ba}_i(v) := \sum_{T \subseteq N, i \in T} \frac{\lambda_{T}(v)}{2^{|T|-1}} \quad \text{for all } i \in N, \ v \in \mathbb{V}.
\]

### 3. Efficient, linear, and symmetric values

In this section, we first provide the formulae for the class of efficient, linear, and symmetric values (henceforth, **ELS values**) mentioned in the introduction. The formulae below apply to all \( v \in \mathbb{V} \) and \( i \in N \).

**Ruiz et al. (1998)**: For \( \rho = (\rho_1, \ldots, \rho_{|N|-1}) \in \mathbb{R}^{|N|-1} \), the ELS value \( \text{RVZ}^\rho \) is given by
\[
\text{RVZ}_i^\rho(v) := \frac{v(N)}{|N|} + \sum_{S \subseteq N, i \in S} \frac{\rho_s}{|S|} \cdot v(S) - \sum_{S \subseteq N \setminus \{i\} : S \neq \emptyset} \frac{\rho_s}{|N| - |S|} \cdot v(S).
\]

**Driessen and Radzik (2003)**: For \( b = (b_1, \ldots, b_{|N|-1}) \in \mathbb{R}^{|N|-1} \), the ELS value \( \text{DR}^b \) is given by
\[
\text{DR}_i^b(v) := \frac{v(N)}{|N|} + \sum_{S \subseteq N \setminus \{i\}} \frac{b_{|S|+1}}{|N|} \cdot v(S \cup \{i\}) - \sum_{S \subseteq N \setminus \{i\} : S \neq \emptyset} \frac{b_{|S|}}{|N|} \cdot v(S) \quad \text{(4)}
\]

A major disadvantage of the above formulae is that the parameters can hardly be interpreted in economic terms. To remedy this, Chameni-Nembua (2012) proposes another type of parametrization\(^4\) For \( \alpha = (\alpha_2, \ldots, \alpha_{|N|}) \in \mathbb{R}^{|N|-1} \), the ELS value \( \text{CN}^\alpha \) is given by
\[
\text{CN}_i^\alpha(v) := \frac{v(\{i\})}{|N|} + \sum_{S \subseteq N, i \in S, |S| > 1} \frac{\text{AMC}_i^\alpha(S, \alpha)}{|N|} \cdot \frac{1}{|S|}.
\]

\(^4\)Chameni-Nembua and Andjiga (2008) and Malawski (2013) and personal communication) consider essentially the same formulae, the latter under the name **inversely procedural values**. Moreover, Hernandez-Lamoneda et al. (2008) consider similar parametrizations, which just rescale the parameters. Actually, they consider continuous values and require just additivity. Yet, it is well-known that linearity entails continuity and that additivity combined with continuity implies linearity.

\(^4\)Malawski (2013) suggests essentially the same formulae as the **procedural values**. Instead of marginal contributions to coalitions, he considers marginal contributions for orders of the player set.
where

$$AMC_i^p (S, \alpha) := \alpha_{|S|} \cdot [v(S) - v(S \setminus \{i\})] + \frac{1 - \alpha_{|S|}}{|S| - 1} \sum_{j \in S \setminus \{i\}} [v(S) - v(S \setminus \{j\})].$$

According to this formula, a player’s payoff is some average of marginal contributions, both of his own ones and the other player ones. Within a coalition $S$, the marginal contribution of player $i \in S$ is taxed at a rate of $1 - \frac{\alpha_{|S|}}{|S| - 1}$, leaving him a share of $\alpha_{|S|} \cdot [v(S) - v(S \setminus \{i\})]$, while the tax revenue amounting to $(1 - \alpha_{|S|}) \cdot [v(S) - v(S \setminus \{i\})]$ is distributed evenly among the other players in $S$.

Despite of the structural differences of the formulae above, they are closely related. By applying these values to standard games, the parameters can be recovered in a similar fashion. In particular, for $T \not\subseteq N$, $T \neq \emptyset$, and $i \in T$, we have

$$\rho_{|T|} = |T| \cdot RVZ_i^p (e_T),$$
$$b_{|T|} = \left( \frac{|N|}{|T|} \right) \cdot |T| \cdot DR_i^b (e_T),$$
$$\alpha_{|T|+1} = \left( \frac{|N|}{|T|} \right) \cdot |T| \cdot CN_i^a (e_T).$$

Hence, conditions on the parameters as for example imposed by [Radzik and Driessen (2013)] for the formula suggested by [Driessen and Radzik (2003)] can easily be translated into conditions for the parameters of the other formulae above.

We advocate another formula for the class of ELS values, already indicated by [Radzik and Driessen (2009), p. 5]. In contrast to the approaches above, our formula is based on to unanimity games, i.e., on the Harsanyi dividends $\lambda_T (v)$ in $[1]$. We consider the following class of values on $N$. For $\tau = (\tau_1, \ldots, \tau_{|N|-1}) \in \mathbb{R}^{|N|-1}$, the value $\zeta^\tau$ on $N$ is given by

$$\zeta^\tau (v) = \frac{\lambda_N (v)}{|N|} + \sum_{T \subseteq N : T \neq \emptyset} \frac{\tau_{|T|}}{|N|} \cdot \lambda_T (v) + \sum_{T \subseteq N : \exists i \in T} \left( 1 - \frac{\tau_{|T|}}{|T|} \right) \cdot \lambda_T (v),$$

for all $i \in N$, $v \in V$.

While the formulae in the previous section are closely related via (5), our formula is distinct. Instead of standard games, unanimity games are employed to recover the parameters. For $T \not\subseteq N$, $T \neq \emptyset$, and $i \in N \setminus T$, we have

$$\tau_{|T|} = |N| \cdot \zeta^\tau_i (u_T).$$

The parameters $(\tau_1, \ldots, \tau_{|N|-1})$ can be interpreted as tax rates that are applied to (scaled) unanimity games. For $\lambda \cdot u_T$, $\lambda \in \mathbb{R}$, $T \subseteq N$, $T \neq \emptyset$, we obtain

$$\zeta^\tau_i (\lambda \cdot u_T) = \frac{\tau_{|T|} \cdot Sh_N (\lambda \cdot u_T)}{|N|} + (1 - \tau_{|T|}) \cdot Sh_i (\lambda \cdot u_T)$$

for all $i \in N$. 5
That is, player $i$’s Shapley payoff is taxed at a rate of $\tau_{|T|}$, leaving him a net income of $\tau_{|T|} \cdot Sh_i(\lambda \cdot u_T)$, while the resulting overall tax revenue amounting to $\tau_{|T|} \cdot Sh_N(\lambda \cdot u_T)$ is distributed evenly among all players. Note that this kind of taxation and redistribution would not affect the payoffs for $\lambda \cdot u_N$. Hence, there is no tax rate $\tau_{|N|}$. The following proposition is immediate from (6).

Proposition 1. A value $\varphi$ on $N$ satisfies $L$, $E$, and $S$ if and only if there is some $\tau \in \mathbb{R}^{|N|-1}$ such that $\varphi = \zeta^\tau$, where $\zeta^\tau$ is as in (7).

A number of values in the literature belong to the class of ELS values. In Table 1, we provide the tax rates $\tau \in \mathbb{R}^{|N|-1}$ for some of them. Unfortunately, there seems to be no “nice” expressions for the tax rates that produce the solidarity value.

4. Solidarity and fair taxation

Within the class of ELS values dwells a huge number of values that do not show certain economically sound properties. In this section, we provide conditions on the parameters of our formula (6) such that one or another of the desirable properties mentioned in the introduction is satisfied. These properties can be viewed as requirements of fair taxation.

4.1. Technical preliminaries

Later on, we will make use of the following definitions. For $m \in \mathbb{N}$ and $x \in \mathbb{R}^m$, the forward differences $\Delta_t^k x$, $t \in \{1, \ldots, m\}$, $k \in \{0, \ldots, m - t\}$ are given recursively by

$$
\Delta_t^0 x := x_t \quad \text{and} \quad \Delta_t^{k+1} x := \Delta_t^k x - \Delta_{t+1}^k x \quad \text{for all} \ t \in \{1, \ldots, m\}, \ k \in \{0, \ldots, m - t\}.
$$

It is well-known that $\Delta_t^k x$ is given by

$$
\Delta_t^k x = \sum_{\ell=0}^{k} (-1)^{\ell} \cdot \binom{k}{\ell} \cdot x_{t+\ell} \quad \text{for all} \ t \in \{1, \ldots, m\}, \ k \in \{0, \ldots, m - t\}.
$$

The following lemma drops from (8) by induction on $t + k$. 

|     | $\tau_1$ | $\tau_2$ | $\ldots$ | $\tau_t$ | $\ldots$ | $\tau_{|N|-1}$ |
|-----|----------|----------|----------|----------|----------|----------------|
| Sh  | 0        | 0        | $\ldots$ | 0        | $\ldots$ | 0              |
| $Sh^*$ | $1 - \alpha$ | $1 - \alpha$ | $\ldots$ | $1 - \alpha$ | $\ldots$ | $1 - \alpha$ |
| CON | 0        | $\frac{1}{2}$ | $\ldots$ | $\frac{1}{2}$ | $\ldots$ | $\frac{1}{2}$ |
| ES  | 0        | 1        | $\ldots$ | 1        | $\ldots$ | 1              |
| LSPN| $1 - \frac{1}{|N|}$ | $1 - \frac{1}{|N|}$ | $\ldots$ | $1 - \frac{t}{2^{|N|-2}} \frac{1}{|N|}$ | $\ldots$ | $1 - \frac{|N| - 1}{2^{|N|-2}} \frac{1}{|N|}$ |
| ED  | 1        | 1        | $\ldots$ | 1        | $\ldots$ | 1              |

Table 1: Tax rates for some ELS values
Lemma 2. Let $m \in \mathbb{N}$ and $x \in \mathbb{R}^m$. Then, $\Delta_t^{m-t} x \geq 0$ for all $t \in \{1, \ldots, m\}$ implies $\Delta_t^k x \geq 0$ for all $t \in \{1, \ldots, m\}$, $k \in \{0, \ldots, m-t\}$.

Moreover, we employ a transformation of $x \in \mathbb{R}^m$, $m \in \mathbb{N}$. We consider $[x] \in \mathbb{R}^m$ defined by

$$[x]_t := \frac{x_t}{t} \quad \text{for all } t \in \{1, \ldots, m\}.$$  \hfill (10)

Let $0, 1 \in \mathbb{R}^m$ be given by $0_t = 0$ and $1_t = 1$ for all $t \in \{1, \ldots, m\}$. By induction on $k$, one easily shows

$$\Delta_t^k [\rho \cdot 1] = \frac{\rho}{(t+k) \binom{t+k-1}{k}}$$  \hfill (11)

for all $\rho \in \mathbb{R}$, $t \in \{1, \ldots, m\}$, and $k \in \{0, \ldots, m-t\}$.

4.2. Relation between parameters

We prepare our main results by establishing the relation between our parameters and the parameters of the other formulae for ELS values. In view of (5), we focus on the formula suggested by Driessen and Radzik (2003). First, we show how our parameters can be translated into parameters for the latter formula. The proof of the proposition is referred to the appendix.

Proposition 3. For $b \in \mathbb{R}^{|N|\!-\!1}$ and $\tau \in \mathbb{R}^{|N|\!-\!1}$, we have $\text{DR}^b = \zeta^\tau$ if and only if

$$b_t = 1 - \frac{\Delta_t^{N\!-\!1\!-\!t} [\tau]}{\Delta_t^{N\!-\!1\!-\!t} [1]} \quad \text{for all } t \in \{1, \ldots, |N| - 1\}.$$ 

Now, we show how the parameters of the formula suggested by Driessen and Radzik (2003) can be translated into the parameters of our formula. The proof of the proposition is referred to the appendix.

Proposition 4. For $b \in \mathbb{R}^{|N|\!-\!1}$ and $\tau \in \mathbb{R}^{|N|\!-\!1}$, we have $\text{DR}^b = \zeta^\tau$ if and only if

$$\tau_t = 1 - \frac{1}{\binom{|N| - 1}{t}} \sum_{s=t}^{N-1} \binom{s-1}{t-1} \cdot b_s \quad \text{for all } t \in \{1, \ldots, |N| - 1\}.$$ 

4.3. Desirability

Even if players express solidarity among themselves, the payoffs should reflect their individual productivity. At least, payoff differentials should not be opposite to their productivities. This idea is expressed by the desirability axiom.

Desirability, $D$ (Maschler and Peleg, 1966). For all $v \in \mathbb{V}$ and $i, j \in N$ such that $MC_i^v (S) \geq MC_j^v (S)$ for all $S \subseteq N \setminus \{i, j\}$, we have $\varphi_i (N, v) \geq \varphi_j (N, v)$.\footnote{Desirability is also known as local monotonicity (e.g. Levinský and Silársky 2004) or fair treatment (e.g. Radzik and Driessen 2013).}
The ELS value $\text{DR}^b$ in (4) meets desirability if and only if $b_t \geq 0$ for all $t \in \{1, \ldots, |N| - 1\}$ \cite{Radzik and Driessen 2013}. Combining this result with Proposition 3, we obtain the following requirements on our parameters to guarantee desirability. Casajus (2012) provides a direct proof of this result.

\textbf{Theorem 5.} The ELS value $\zeta^\tau, \tau \in \mathbb{R}^{|N|-1}$ obeys desirability (D) if and only if $\Delta_i^{|N|-1-t}[1] \geq \Delta_i^{|N|-1-t}[^\tau]$ for all $t \in \{1, \ldots, |N| - 1\}$.

\textbf{Remark 6.} Theorem 5 together with Lemma 2 implies the following necessary requirements on $\tau \in \mathbb{R}^{|N|-1}$ for $\zeta^\tau$ to satisfy desirability: $\Delta_i^k[1] \geq \Delta_i^k[^\tau]$ for all $t \in \{1, \ldots, |N| - 1\}$ and $k \in \{0, \ldots, |N| - t - 1\}$. In some strong sense, taxes should be smaller than 1. In particular, we have (i) $\Delta_i^1[1] \geq \Delta_i^1[^\tau], \text{i.e., } 1 \geq \tau_t$ for all $t \in \{1, \ldots, |N| - 1\}, \text{i.e., the players should not be overtaxed. Further, (ii) } \Delta_i^1[1] \geq \Delta_i^1[^\tau], \text{i.e., } \tau_{t+1} \geq \tau_t - \frac{1-\tau_t}{t}$ for all $t \in \{1, \ldots, |N| - 2\}$. Given $1 \geq \tau_t$, this means that tax rates should not decrease too much when $t$ increases. In particular, if $\tau_t = 1$ for some $t$, then $\tau_s = 1$ for all $s \geq t$.

4.4. Positivity for null players

In monotonic games, no player ever is destructive, i.e., all players always have a non-negative marginal contributions. Hence, even if players show solidarity to less productive ones, nobody should end up with a sub-zero payoff. This idea is expressed by the positivity axiom.

\textbf{Positivity} \cite{Kalai and Samet 1987}, P. For all $v \in V$ that are monotonic and all $i \in N$, we have $\varphi_i(N, v) \geq 0$ \cite{Radzik and Driessen 2013}. Radzik and Driessen \cite{2013} do not give exact conditions on $b \in \mathbb{R}^{|N|-1}$ such that the value $\text{DR}^b$ meets positivity. Careful inspection of the proof of their Theorem 2 shows that one actually has a nice description of those $b \in \mathbb{R}^{|N|-1}$ that let $\text{DR}^b$ obey a weaker property, positivity for null players.

\textbf{Positivity for null players}, PN. For all $v \in V$ that are monotonic and all $i \in N$ who are null players in $v$, we have $\varphi_i(N, v) \geq 0$.

\textbf{Theorem 7.} The value $\text{DR}^b, b \in \mathbb{R}^{|N|-1}$ obeys positivity for null players (PN) if and only if $1 \geq b_t$ for all $t \in \{1, \ldots, |N| - 1\}$.

\textbf{Proof.} Sufficiency: Let $v \in V$ be monotonic and $i \in N$ be a null player in $v$. Let further $b \in \mathbb{R}^{|N|-1}$ be such that $1 \geq b_t$ for all $t \in \{1, \ldots, |N| - 1\}$. This part of the proof heavily relies on the proof of Radzik and Driessen \cite{2013} (Theorem 2), which makes use of the additional assumption (*) $b_t \geq 0$ for all $t \in \{1, \ldots, |N| - 1\}$. Since $i$ is a null player in $v$, we can avoid to appeal to (*). For notational parsimony, we just indicate how this can be done. Condition (*) is applied twice, first, in the proof of the induction basis, and second, in the final step of the proof.

\footnote{Positivity is also known as monotonicity \cite[e.g.][]{Radzik and Driessen 2013}.}
By (4), we have

$$\text{DR}_t^b (v) = \frac{v(N)}{|N|} + \sum_{S \subseteq N \setminus \{i\}} \frac{b_{|S|+1} \cdot v(S \cup \{i\})}{|N|} - \sum_{S \subseteq N \setminus \{i\} : S \neq \emptyset} \frac{b_{|S|} \cdot v(S)}{|N| + 1} \cdot (|S| + 1)$$

$$= (1 - b_{|N|-1}) \cdot \frac{v(N)}{|N|} + \sum_{S \subseteq N \setminus \{i\}} \frac{(b_{|S|+1} - b_{|S|}) \cdot v(S \cup \{i\})}{|N|} \cdot (|S| + 1),$$

where the second equation follows from being a null player. Yet, this already is the induction basis. Since $v(\{i\}) = 0$ for the null player $i$, we do not need (*) in the final step of the proof.

Necessity: Let now $b \in \mathbb{R}^{|N|-1}$ be such that $\text{DR}_t^b$ meets PN. Fix $t \in \{1, \ldots, |N| - 1\}$ and $i \in N$ and let $v \in \mathbb{V}$ be given by $v(S) = 1$ if $|S| > t$ and $i \in S$, $v(S) = 1$ if $|S| = t$ and $i \notin S$, and $v(S) = 0$ else. One easily checks that $v$ is monotonic and that $i$ is a null player in $v$. Moreover, Radzik and Driessen (2013, Proof of Theorem 2) show $\text{DR}_t^b (v) = \frac{1 - b_t}{|N|}$. Hence, $b_t \leq 1$ for all $t \in \{1, \ldots, |N| - 1\}$.

Combining this result with Proposition 3, we obtain the following requirements on our parameters to guarantee desirability. Casajus (2012) provides a direct proof of this result.

**Theorem 8.** The value $\zeta^\tau$, $\tau \in \mathbb{R}^{|N|-1}$ obeys positivity for null players (PN) if and only if $\Delta^{[N]-1-t}_t [\tau] \geq \Delta^{[N]-1-t}_t [0] = 0$ for all $t \in \{1, \ldots, |N| - 1\}$.

**Remark 9.** Theorem 8 together with Lemma 2 implies the following necessary requirements on $\tau \in \mathbb{R}^{|N|-1}$ for $\zeta^\tau$ to satisfy positivity for null players: $\Delta^k_t [\tau] \geq \Delta^k_t [0] = 0$ for all $t \in \{1, \ldots, |N| - 1\}$ and $k \in \{0, \ldots, |N| - t - 1\}$. In some strong sense, taxes should be non-negative. In particular, we have (i) $\Delta^0_t [\tau] \geq 0$, i.e., $\tau_t \geq 0$ for all $t \in \{1, \ldots, |N| - 1\}$, i.e., the players should not be subsidized. Further, (ii) $\Delta^1_t [\tau] \geq 0$, i.e., $\frac{t+1}{t} \tau_t \geq \tau_{t+1}$ for all $t \in \{1, \ldots, |N| - 2\}$. Given $\tau_t, \tau_{t+1} \geq 0$, this means that tax rates should not increase too much when $t$ increases. In particular, if $\tau_t = 0$ for some $t$, then $\tau_s = 0$ for all $s \geq t$.

**Remark 10.** Casajus and Huettner (2013) consider a considerable sharpening of positivity for null players, the null player in a productive environment property below. Instead of restricting attention to monotonic games, they extend the implication of PN to games where the grand coalition generates a non-negative worth.

**Null player in a productive environment, NPE.** For all $v \in \mathbb{V}$ and $i \in N$ such that $i$ is a null player in $v$ and $v(N) \geq 0$, we have $\varphi_i (v) \geq 0$.

Their Proposition 1 entails that the value $\zeta^\tau$, $\tau \in \mathbb{R}^{|N|-1}$ obeys the null player in a productive environment property if and only if $\tau_t = \tau_{t+1} \geq 0$ for all $t \in \{1, \ldots, |N| - 1\}$. By (11) and Proposition 3, the values $\text{DR}_t^b$, $b \in \mathbb{R}^{|N|-1}$ satisfy this property if and only if $1 \geq b_t = b_1$ for all $t \in \{1, \ldots, |N| - 1\}$. 

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4.5. Desirability and positivity

The ELS value $DR^b$, $b \in \mathbb{R}^{[N]-1}$ satisfies desirability and positivity if and only if $1 \geq b_t \geq 0$ for all $t \in \{1, \ldots, |N| - 1\}$ ([Radzik and Driessen] 2013, Theorem 2). Combining this result with Proposition 3, we obtain the following requirements on our parameters to guarantee for all $(\tau)$

**Theorem 11.** The value $\zeta^*$, $\tau \in \mathbb{R}^{[N]-1}$ obeys desirability (D) and positivity (P) if and only if $\Delta_t^{[N]-1-t}[1] \geq \Delta_t^{[N]-1-t}[\tau] \geq \Delta_t^{[N]-1-t}[0] = 0$ for all $t \in \{1, \ldots, |N| - 1\}$.

**Remark 12.** Theorem 11 together with Lemma 2 implies the following necessary requirements on $\tau \in \mathbb{R}^{[N]-1}$ for $\zeta^*$ to satisfy desirability and positivity for null: $\Delta_t^k[1] \geq \Delta_t^k[\tau] \geq \Delta_t^k[0] = 0$ for all $t \in \{1, \ldots, |N| - 1\}$ and $k \in \{0, \ldots, |N| - t - 1\}$. In a strong sense, tax rates are required to fall between 0 and 1.

**Remark 13.** Theorems 5, 8, and 11 alternatively, Theorem 7 and [Radzik and Driessen] (2013, Theorems 1 and 2) imply that an ELS value that satisfies desirability and positivity for null players also satisfies positivity. Casajus (2012) provides a direct proof of this result.

We now demonstrate the power of Theorem 11 with an example. A technical lemma facilitates the application of the theorem.

**Lemma 14.** Let $m \in \mathbb{N}$ and $f : [1, m] \to \mathbb{R}$ be differentiable up to order $m-1$ and such that $(-1)^k \cdot f^{(k)}(\xi) \geq 0$ for all $\xi \in [1, m]$ and $k \in \{0, \ldots, m-1\}$. For $x \in \mathbb{R}^m$ given by $x_t = f(t)$ for all $t \in \{1, \ldots, m\}$, we have $\Delta_t^k x \geq 0$ for all $t \in \{1, \ldots, m\}$ and $k \in \{0, \ldots, m-t\}$.

**Proof.** Let $m$ and $f$ be as in the lemma. For $t \in \{1, \ldots, m\}$, we have

$$\Delta_t^0 x = x_t = f(t) = f^{(0)}(t) = (-1)^0 \cdot f^{(0)}(t) \geq 0.$$

By induction on $k$, one easily shows

$$\Delta_t^k x = (-1)^k \int_t^{t+1} \int_i^{i+1} \int_j^{j+1} \cdots \int_k^{k+1} f^{(k)}(\xi) d\xi_k \cdots d\xi_3 d\xi_2$$

for all $t \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, m-t\}$. The claim now follows from $(-1)^k \cdot f^{(k)}(\xi) \geq 0$ for all $\xi \in [1, m]$.

**Example 15.** For $\alpha \in [0, 1]$, we consider the tax system $\tau^\alpha \in \mathbb{R}^{[N]-1}$ such that

$$\zeta^\alpha_i (u_T) = \alpha \cdot \zeta^\alpha_j (u_T), \quad \text{for all } T \subset N, \ T \neq \emptyset, \ i \in N \ T, \ j \in T.$$ 

That is, in unanimity games, unproductive players obtain $\alpha$ times the payoff of productive players. By (6), we obtain

$$\tau^\alpha_t = \frac{\alpha \cdot |N|}{(1-\alpha) \cdot t + \alpha \cdot |N|}$$

for all $t \in \{1, \ldots, |N| - 1\}$. 

The resulting value $\zeta^\tau$ meets $D$ and $P$. To see this, let $f, g : [1, |N| - 1] \to \mathbb{R}$ be given by

$$f(\xi) = \frac{1}{\xi}, \quad g(\xi) = \frac{1 - \alpha}{(1 - \alpha) \cdot \xi + \alpha \cdot |N|}$$

for all $\xi \in [1, |N| - 1]$.

By (10), we have $[\tau^\alpha]_t = f(t) - g(t)$ and $([1] - [\tau^\alpha])_t = g(t)$ for $t \in \{1, \ldots, |N| - 1\}$. Moreover, one obtains $f^{(0)}(\xi) \geq g^{(0)}(\xi) \geq 0$, and

$$(-1)^k \cdot f^{(k)}(\xi) = \frac{(-1)^{2k} \cdot k!}{\xi^{k+1}} \geq \frac{(-1)^{2k} \cdot k! \cdot (1 - \alpha)^{k+1}}{((1 - \alpha) \cdot \xi + \alpha \cdot |N|)^{k+1}} = (-1)^k \cdot g^{(k)}(\xi) \geq 0$$

for all $\xi \in [1, |N| - 1]$, $k \in \{0, \ldots, |N| - 2\}$. By Lemma 14, we have $\Delta_{[1]}^{[N] - 1 - t}[1] \geq \Delta_{[N] - 1 - t}^{[N] - 1 - t}[\tau^\alpha] \geq 0$ for all $t \in \{1, \ldots, |N| - 1\}$. Finally, the claim follows from Theorem 11.

4.6. Social acceptability

Joosten et al. (1994) consider the social acceptability axiom.

Social acceptability, SA. For all $T \subseteq N$, $i \in T$, and $j \in N \setminus T$, we have $\varphi_i(u_T) \geq \varphi_j(u_T) \geq 0$.

Social acceptability imposes rather weak fairness requirements. Since unanimity games are monotonic, the requirement $\varphi_i(u_T) \geq 0$ and $\varphi_j(u_T) \geq 0$ above is equivalent to positivity restricted to unanimity games. In $u_T$, the players in $T$ are more productive than those in $N \setminus T$. Hence for ELS values, demanding $\varphi_i(u_T) \geq \varphi_j(u_T)$ for $i \in T$ and $j \in N \setminus T$ is equivalent to desirability restricted to unanimity games.

Since the values $\zeta^\tau$ are closely related to the linear representation of games by unanimity games, we state the following obvious proposition with some diffidence and mainly for completeness’ sake. Economically, it just says that there should be no subsidizing and no overtaxing of productive players in unanimity games.

Proposition 16. The value $\zeta^\tau$, $\tau \in \mathbb{R}^{[N] - 1}$ obeys social acceptability (SA) if and only if $1 \geq \tau_t \geq 0$ for all $t \in \{1, \ldots, |N| - 1\}$.

Proof. Fix $T \subseteq N$, $|T| = t < |N|$. Let $i \in T$, $j \in N \setminus T$. By (6), we have $\zeta^\tau_i(u_T) - \zeta^\tau_j(u_T) = \frac{1 - \tau_t}{|T|} \geq 0$ if and only if $\tau_t \leq 1$ and $\zeta^\tau_j(u_T) = \frac{T}{|N|} \geq 0$ if $\tau_t \geq 0$. Further, $\zeta^\tau_i(u_N) = |N|^{-1} > 0$ for all $i \in N$.

Remark 17. Compare the results of the proposition with analogous findings for the formulae based on standard games. The ELS value $\text{DR}^b$, $b \in \mathbb{R}^{[N] - 1}$ satisfies social acceptability if and only if

$$0 \leq \frac{|N| \cdot t}{|N| - t} \cdot \left(\frac{|N|}{t}\right)^{-1} \cdot \sum_{s=t}^{[N] - 1} \binom{s}{t} \cdot \frac{b_s}{s} \leq 1$$

for all $t \in \{1, \ldots, |N| - 1\}$ (Radzik and Driessen 2013, Theorem 3).
4.7. General acceptability

Radzik and Driessen (2013) consider another notion of acceptability, general acceptability.

**General acceptability, GA.** For all $S, T \subseteq N$ and $i \in N$ such that $S \subseteq T$ and $i \in S$, we have $\varphi_i(u_S) \geq \varphi_i(u_T)$.

Within the class of ELS values, general acceptability coincides with strong monotonicity for unanimity games. Note that on the domain of all TU games, there is a unique ELS value that meets strong monotonicity, the Shapley value (Young, 1985, Theorem 2).

**Strong monotonicity, Mo** (Young, 1985). For all $v, w \in V$ and $i \in N$ such that $v(K \cup \{i\}) - v(K) \geq w(K \cup \{i\}) - w(K)$ for all $K \subseteq N \setminus \{i\}$, we have $\varphi_i(v) \geq \varphi_i(w)$.

**Proposition 18.** The value $\zeta^\tau$, $\tau \in \mathbb{R}^{[N]-1}$ obeys general acceptability (GA) if and only if

- (i) $\tau_t \leq 1$ for all $t \in \{1, \ldots, |N| - 1\}$ and (ii)
  \[
  \tau_{t+1} - \tau_t \geq \frac{\tau_t - 1}{t} \cdot \frac{|N|}{|N| - t - 1}
  \]

for all $t \in \{1, \ldots, |N| - 2\}$.

**Proof.** Let $t \in \{1, \ldots, |N| - 1\}$ and $T \subseteq N$, $|T| = t$. By (6), $\zeta^\tau(T) \geq \zeta^\tau(N)$ iff $\tau_t \leq 1$. Let $s \in \{1, \ldots, |N| - 2\}$ and $S, T \subseteq N$, $S \subseteq T$, $|S| = s$, $|T| = s + 1$. By (6), $\zeta^\tau(S) \geq \zeta^\tau(T)$ if and only if

\[
\tau_{s+1} \geq \tau_s - \frac{1 - \tau_s}{s} \cdot \frac{|N|}{|N| - s - 1},
\]

which entails the second part of the requirement. \qed

**Remark 19.** Proposition 18 first requires that there is no overtaxing, $\tau_t \leq 1$. Given this, the second requirement says that tax rates should not decrease too much when $t$ increases. In particular, if $\tau_t = 1$ for some $t$, then $\tau_s = 1$ for all $s \geq t$. Recall some necessary requirements for desirability due to Theorem 5 (i) $\tau_s \leq 1$ for all $t \in \{1, \ldots, |N| - 1\}$ and (ii) $\tau_{t+1} \geq \tau_t - \frac{1 - \tau_t}{t}$ for all $t \in \{1, \ldots, |N| - 2\}$. Since $\tau_t - 1 \leq 0$ and $\frac{|N|}{|N| - t - 1} > 1$, desirability implies general acceptability for ELS values.

**Remark 20.** Compare the results of the proposition with analogous findings for the formulae based on standard games. The ELS value $\text{DR}^b$, $b \in \mathbb{R}^{[N]-1}$ satisfies general acceptability if and only if

\[
0 \leq \sum_{s=t}^{[N]-1} \frac{|N| - s}{s} \cdot \binom{s}{t} \cdot b_s
\]

for all $t \in \{1, \ldots, |N| - 1\}$ Radzik and Driessen (2013, Theorem 4).
Appendix: Omitted proofs

Proof of Proposition 3

Necessity: Let \( S \subset N \), \( S \neq \emptyset \), \( i \in S \) and \( j^* \in N \setminus S \). First, we determine the payoff \( \zeta^i_T(es) \). It is well-known that

\[
\lambda_T(es) = \begin{cases} 
0, & S \not\subseteq T, \\
(-1)^{|T|-|S|}, & S \subseteq T
\end{cases} \text{ for all } T \subseteq N. \tag{12}
\]

We obtain

\[
\zeta^i_T(es) = \frac{\lambda_N(es)}{|N|} + \sum_{T \subseteq N:T \neq \emptyset} \frac{\tau_T}{|N|} \cdot \lambda_T(es) + \sum_{T \subseteq N: i \in T} \left( \frac{1 - \tau_T}{|T|} \right) \cdot \lambda_T(es)
\]

\[
= \frac{\lambda_N(es)}{|N|} + \frac{1}{|N|} \sum_{j \in N} \sum_{j \in T \subseteq N} \frac{\tau_T}{|T|} \cdot \lambda_T(es) + \sum_{T \subseteq N: i \in T} \frac{1 - \tau_T}{|T|} \cdot \lambda_T(es)
\]

\[
= \sum_{S \subseteq T \subseteq N} \frac{(-1)^{|N|-|S|}}{|N|} + \frac{1}{|N|} \sum_{j \in N} \sum_{S \subseteq T \subseteq N} \frac{\tau_T}{|T|} \cdot (-1)^{|T|-|S|} + \sum_{S \subseteq T \subseteq N} \frac{1 - \tau_T}{|T|} \cdot (-1)^{|T|-|S|}
\]

\[
\sum_{j^* \in N \setminus S} \sum_{S \subseteq T \subseteq N} \frac{(-1)^{|N|-|S|}}{|N|} - \frac{|N| - |S|}{|N|} \sum_{S \subseteq T \subseteq N} \frac{\tau_T}{|T|} \cdot (-1)^{|T|-|S|}
\]

\[
= \sum_{S \subseteq T \subseteq N} \frac{|N| - |S|}{|N|} \left( \frac{|N| - |S|}{t - |S|} \right) \cdot (-1)^{t-|S|} - \frac{|N| - |S|}{|N|} \sum_{t=|S|}^{N-1} \left( \frac{|N| - |S|}{t - |S|} \right) \cdot \frac{\tau_t}{t} \cdot (-1)^{t-|S|}
\]

\[
+ \frac{|N| - |S|}{|N|} \sum_{t=|S|}^{|N|-2} \left( \frac{|N| - 1 - |S|}{t - |S|} \right) \cdot \frac{\tau_{t+1}}{t+1} \cdot (-1)^{t+1-|S|}
\]

\[
= \sum_{t=0}^{|N|-|S|} \left( \frac{|N| - |S|}{t} \right) \cdot \frac{(-1)^t}{t + |S|} - \frac{|N| - |S|}{|N|} \sum_{t=0}^{|N|-|S|} \left( \frac{|N| - |S|}{t} \right) \cdot \frac{\tau_{t+1}}{t + |S|} \cdot (-1)^t
\]

\[
+ \frac{|N| - |S|}{|N|} \sum_{t=0}^{|N|-2} \left( \frac{|N| - 1 - |S|}{t} \right) \cdot \frac{\tau_{t+1+|S|}}{t + 1 + |S|} \cdot (-1)^{t+1}. \tag{13}
\]
Moreover, we have
\[
\sum_{t=0}^{\left|N\right|-\left|S\right|} \binom{\left|N\right|-\left|S\right|}{t} \cdot \frac{\tau_{t+\left|S\right|}}{t+\left|S\right|} \cdot (-1)^t \\
= \left(\binom{\left|N\right|-\left|S\right|}{0}\right) \frac{\tau_{0+\left|S\right|}}{0+\left|S\right|} \cdot (-1)^0 + \sum_{t=1}^{\left|N\right|-1-\left|S\right|} \left(\binom{\left|N\right|-1-\left|S\right|}{t}\right) \cdot \frac{\tau_{t+\left|S\right|}}{t+\left|S\right|} \\
+ \sum_{t=1}^{\left|N\right|-1-\left|S\right|} \left(\binom{\left|N\right|-1-\left|S\right|}{t-1}\right) \cdot \frac{\tau_{t+\left|S\right|}}{t+\left|S\right|} = \sum_{t=0}^{\left|N\right|-1-\left|S\right|} \left(\binom{\left|N\right|-1-\left|S\right|}{t}\right) \cdot \frac{\tau_{t+\left|S\right|}}{t+\left|S\right|} + \sum_{t=0}^{\left|N\right|-2-\left|S\right|} \left(\binom{\left|N\right|-1-\left|S\right|}{t}\right) \cdot \frac{\tau_{t+1+\left|S\right|}}{t+1+\left|S\right|}.
\] (14)

Combining (13) and (14) gives
\[
\zeta^t_e(S) = \sum_{t=0}^{\left|N\right|-\left|S\right|} \left(\binom{\left|N\right|-\left|S\right|}{t}\right) \frac{(-1)^t}{t+\left|S\right|} - \frac{\left|N\right|-\left|S\right|}{\left|N\right|} \sum_{t=0}^{\left|N\right|-1-\left|S\right|} \left(\binom{\left|N\right|-1-\left|S\right|}{t}\right) \frac{\tau_{t+\left|S\right|}}{t+\left|S\right|} \\
= \frac{1}{\left|S\right|} \cdot \frac{\left|N\right|}{\left|S\right|} \Delta^{\left|N\right|-1-\left|S\right|} \left[\tau\right] \\
= \frac{1}{\left|S\right|} \cdot \frac{\left|N\right|}{\left|S\right|} \left(1 - \frac{\Delta^{\left|N\right|-1-\left|S\right|} \left[\tau\right]}{\Delta^{\left|N\right|-1-\left|S\right|} \left[1\right]}\right).
\]

In view of (5), the claim now is immediate. Sufficiency drops from the fact that both formulae cover the class of ELS values. \qed
Proof of Proposition

Necessity: Let $T \not\subseteq N$, $T \neq \emptyset$, $i \in N \setminus S$. First, we determine the payoff $\text{DR}_i^b(u_T)$. One obtains

$$
\text{DR}_i^b(u_T) = \frac{u_T(N)}{|N|} + \frac{1}{|N|} \sum_{S \subseteq N \setminus \{i\}} \frac{b_{|S|+1} \cdot u_T(S \cup \{i\})}{(|N| + 1)(|S| + 1)} - \frac{1}{|N|} \sum_{S \subseteq N \setminus \{i\}: S \neq \emptyset} \frac{b_{|S|} \cdot u_T(S)}{|S| + 1)}
$$

$$
i \in N \setminus S
$$

$$
= \frac{1}{|N|} + \frac{1}{|N| - |T|} \left( \frac{|N|}{|T|} \right) \sum_{s=|T|}^{N-2} \frac{s}{s} \cdot b_{s+1} - \sum_{s=|T|}^{N-1} \frac{s}{s} \cdot b_s
$$

$$
= \frac{1}{|N|} - \frac{b_{|T|}}{|N| - |T|} \left( \frac{|N|}{|T|} \right) + \frac{1}{|N| - |T|} \sum_{s=|T|+1}^{N-1} \left( \frac{s - 1}{|T|} \right) \cdot b_s
$$

$$
= \frac{1}{|N|} - \frac{1}{|N| - |T|} \left( \frac{|N|}{|T|} \right) \sum_{s=|T|}^{N-1} \frac{s - 1}{|T| - 1} \cdot b_s
$$

In view of (7), the claim is immediate.

Sufficiency drops from the fact that both formulae cover the class of ELS values.

References


