The Myerson value in terms of the link agent form: a technical note

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Abstract

We represent the Myerson (1977) value in terms of the value introduced by Vázquez-Brage, García-Jurado and Carreras (1996) applied to the link agent form (Casajus, 2007) accompanied by some natural coalition structure.

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1. Introduction

Recently, Casajus (2007) characterizes the position value (Meessen, 1988; Borm, Owen and Tijs, 1992) for TU games with a cooperation structure (undirected graph on the player set) in terms of the Myerson value (Myerson, 1977) of some natural modification of the original game—the link agent form. In this note, we show that the Myerson value can be obtained from the link agent form by adding some natural coalition structure (partition of the player set) and applying the value introduced by Vázquez-Brage et al. (1996) for TU games that come with both a cooperation structure and a coalition structure.

The plan of this note is as follows: Basic definitions and notation are given in second section. The third section contains our result.

2. Basic definitions and notation

A **(TU) game** is a pair (N, v) consisting of a non-empty and finite set of players N and the **coalition function** $v \in V(N) := \left\{f : 2^N \to \mathbb{R} | v(\emptyset) = 0\right\}$. Subsets of N are called coalitions, and $v(K)$ is called the worth of coalition K. A **value** is an operator φ that assigns payoff vectors to all games, $\varphi(N, v) \in \mathbb{R}^N$.

An **order of a set** N is a bijection $\sigma : N \to \{1, \ldots, |N|\}$ with the interpretation that i is the $\sigma(i)$th player in σ. The set of these orders is denoted by $\Sigma(N)$. The set of players not after i in σ is denoted by $K_i(\sigma) = \{j : \sigma(j) \leq \sigma(i)\}$. The **marginal contribution** of i in σ is defined as $MC_i^v(\sigma) := v(K_i(\sigma)) - v(K_i(\sigma) \setminus \{i\})$. The **Shapley value** Sh (Shapley, 1953) is defined by

$$\mathrm{Sh}_i(N, v) := |\Sigma(N)|^{-1} \sum_{\sigma \in \Sigma(N)} MC_i^v(\sigma), \quad i \in N. \tag{1}$$

For $K \subseteq N$, we denote by $\varphi_K(N, v, \cdot)$ the sum $\sum_{i \in K} \varphi_i(N, v, \cdot)$.

A **coalition structure** on N is a partition $\mathcal{P} \subseteq 2^N$ of N; a **CS-game** is a game together with a coalition structure, $(N, v, \mathcal{P})$. The elements of $\mathcal{P}$ are referred to as components; $\mathcal{P}(i)$ denotes the component containing player i.A **CS-value** is an operator φ that assigns payoff vectors to all CS-games, $\varphi(N, v, \mathcal{P}) \in \mathbb{R}^N$. For any coalition structure $\mathcal{P}$ on N,

$$\Sigma(N, \mathcal{P}) := \{\sigma \in \Sigma(N) | \forall P \in \mathcal{P} \wedge \forall i, j \in P : |\sigma(i) - \sigma(j)| < |P|\} \tag{2}$$

is the set of orders on N **compatible** with $\mathcal{P}$. The **Owen value** (Owen, 1977) is given by

$$\mathrm{Ow}_i(N,v,\mathcal{P}) := |\Sigma(N,\mathcal{P})|^{-1}\sum_{\sigma\in\Sigma(N,\mathcal{P})} MC_i^v(\sigma), \qquad i\in N. \tag{3}$$

A **cooperation structure** on N is an undirected graph (N,L), $L\subseteq L^N := \{\lambda\subseteq N|\,|\lambda|=2\}$. A typical element, **link**, of L is written as λ. Given any graph (N,L), N splits into (maximal connected) components which constitute the partition $\mathcal{C}(N,L)$ of N; $C_i(N,L)\in\mathcal{C}(N,L)$ denotes the component containing $i\in N$. The restriction of L to $K\subseteq N$ is denoted by $L|_K$,

$$L|_K := \{\lambda\in L|\lambda\subseteq K\}. \tag{4}$$

A **CO-game** (communication situation) (N,v,L) is a game (N,v) together with a cooperation structure (N,L); a CO-value is an operator φ that assigns payoff vectors to all CO-games, $\varphi(N,v,L)\in\mathbb{R}^N$.The **Myerson value** μ (Myerson, 1977) is defined by

$$\mu(N,v,H) := \mathrm{Sh}\left(N,v^L\right), \qquad v^L(K) := \sum_{S\in\mathcal{C}(K,L|_K)} v(S), \qquad K\subseteq N. \tag{5}$$

For any CO-game $G=(N,v,L)$, Casajus (2007) defines its **link agent form** $\mathrm{LAF}(G)=\left(\bar{N},\bar{v},\bar{L}\right)$ as follows:

$$\bar{N} = \bigcup_{i\in N}\bar{N}(i), \qquad \bar{N}(i) := \{(i,\lambda)|\lambda\in L_i\} \tag{6a}$$

$$\bar{L} = \bar{L}^o\cup\bigcup_{i\in N} L^{\bar{N}(i)}, \qquad \bar{L}^o := \{\bar{\imath}\bar{\jmath}|ij\in L\}, \qquad \bar{\imath}\bar{\jmath} := \{(i,ij),(j,ij)\} \tag{6b}$$

$$\bar{v}\left(\bar{K}\right) = v\left(N\left(\bar{K}\right)\right), \qquad N\left(\bar{K}\right) := \left\{i\in N|\bar{N}(i)\cap\bar{K}\neq\emptyset\right\}, \quad \bar{K}\subseteq\bar{N} \tag{6c}$$

Vázquez-Brage et al. (1996) combine the Myerson value and the Owen value into a solution concept, ϖ, for games (N,v) that come with both a coalition structure $(N,\mathcal{P})$ and a cooperation structure (N,L), $\varpi(N,v,L,\mathcal{P})\in\mathbb{R}^N$. This ϖ-value is defined by

$$\varpi(N,v,L,\mathcal{P}) = \mathrm{Ow}\left(N,v^L,\mathcal{P}\right). \tag{7}$$

3. Main result

Consider a communication situation $\Gamma=(N,v,L)$, its link agent form $\bar{\Gamma}=\mathrm{LAF}(\Gamma)=\left(\bar{N},\bar{v},\bar{L}\right)$,and the coalition structure $\mathcal{P}^\Gamma=\left\{\bar{N}(i)\,|i\in N\right\}$.

Theorem For all $i \in N$,

$$\mu_i (N, v, L) = \sum_{(i,\lambda) \in \bar{N}(i)} \varpi_{(i,\lambda)} \left(\bar{N}, \bar{v}, \bar{L}, \mathcal{P}^{\Gamma} \right).$$

Proof. First observe that any $\bar{\sigma} \in \Sigma\left(\bar{N}, \mathcal{P}^{\Gamma}\right)$ uniquely induces some $\bar{\sigma}|_N \in \Sigma(N)$ and $\bar{\sigma}|_{\bar{N}(i)} \in \Sigma\left(\bar{N}(i)\right)$as follows: If $i, j \in N$ are such that $\bar{\sigma}(i, \lambda) < \bar{\sigma}(j, \lambda')$ for all $(i, \lambda) \in \bar{N}(i)$ and $(j, \lambda') \in \bar{N}(j)$, then $\bar{\sigma}|_N (i) < \bar{\sigma}|_N (j)$; $\bar{\sigma}|_{\bar{N}(i)} (i, \lambda) < \bar{\sigma}|_{\bar{N}(i)} (i, \lambda')$ iff $\bar{\sigma}(i, \lambda) < \bar{\sigma}(i, \lambda')$ for all $(i, \lambda), (i, \lambda') \in \bar{N}(i)$. For $\sigma \in \Sigma(N)$ and $\rho \in \Sigma\left(\bar{N}(i)\right)$, we set

$$\Sigma_i\left(\bar{N}, \mathcal{P}^{\Gamma}, \sigma, \rho\right) := \left\{ \bar{\sigma} \in \Sigma\left(\bar{N}, \mathcal{P}^{\Gamma}\right) \middle| \, \bar{\sigma}|_N = \sigma \wedge \sigma|_{\bar{N}(i)} = \rho \right\}. \tag{8}$$

For $\Gamma = (N, v, L)$, we have

$$\begin{aligned}
&\sum_{(i,\lambda) \in \bar{N}(i)} \varpi_{(i,\lambda)} \left(\bar{N}, \bar{v}, \bar{L}, \mathcal{P}^{\Gamma} \right) \\
&\overset{(7)}{=} \sum_{(i,\lambda) \in \bar{N}(i)} \mathrm{Ow}_{(i,\lambda)} \left(\bar{N}, \bar{v}^{\bar{L}}, \mathcal{P}^{\Gamma} \right) \\
&\overset{(3)}{=} \sum_{(i,\lambda) \in \bar{N}(i)} \left| \Sigma\left(\bar{N}, \mathcal{P}^{\Gamma}\right) \right|^{-1} \sum_{\bar{\sigma} \in \Sigma\left(\bar{N}, \mathcal{P}^{\Gamma}\right)} MC^{\bar{v}^{\bar{L}}}_{(i,\lambda)} (\bar{\sigma}) \\
&= \left| \Sigma\left(\bar{N}, \mathcal{P}^{\Gamma}\right) \right|^{-1} \sum_{\bar{\sigma} \in \Sigma\left(\bar{N}, \mathcal{P}^{\Gamma}\right)} \sum_{(i,\lambda) \in \bar{N}(i)} MC^{\bar{v}^{\bar{L}}}_{(i,\lambda)} (\bar{\sigma}) \\
&\overset{(8)}{=} \left| \Sigma\left(\bar{N}, \mathcal{P}^{\Gamma}\right) \right|^{-1} \sum_{\sigma \in \Sigma(N)} \sum_{\rho \in \Sigma\left(\bar{N}(i)\right)} \sum_{\bar{\sigma} \in \Sigma_i\left(\bar{N}, \mathcal{P}^{\Gamma}, \sigma, \rho\right)} \sum_{(i,\lambda) \in \bar{N}(i)} MC^{\bar{v}^{\bar{L}}}_{(i,\lambda)} (\bar{\sigma}) \\
&= \left| \Sigma\left(\bar{N}, \mathcal{P}^{\Gamma}\right) \right|^{-1} \sum_{\sigma \in \Sigma(N)} \sum_{\rho \in \Sigma\left(\bar{N}(i)\right)} \sum_{\bar{\sigma} \in \Sigma_i\left(\bar{N}, \mathcal{P}^{\Gamma}, \sigma, \rho\right)} MC^{v^L}_i (\bar{\sigma}|_N) \\
&= \left| \Sigma\left(\bar{N}, \mathcal{P}^{\Gamma}\right) \right|^{-1} \sum_{\sigma \in \Sigma(N)} \sum_{\rho \in \Sigma\left(\bar{N}(i)\right)} \sum_{\bar{\sigma} \in \Sigma_i\left(\bar{N}, \mathcal{P}^{\Gamma}, \sigma, \rho\right)} MC^{v^L}_i (\sigma) \\
&= \left| \Sigma\left(\bar{N}, \mathcal{P}^{\Gamma}\right) \right|^{-1} \sum_{\sigma \in \Sigma(N)} MC^{v^L}_i (\sigma) \sum_{\rho \in \Sigma\left(\bar{N}(i)\right)} \sum_{\bar{\sigma} \in \Sigma_i\left(\bar{N}, \mathcal{P}^{\Gamma}, \sigma, \rho\right)} 1 \\
&= \frac{\left| \Sigma\left(\bar{N}(i)\right) \right| \left| \Sigma_i\left(\bar{N}, \mathcal{P}^{\Gamma}, \sigma, \rho\right) \right|}{\left| \Sigma\left(\bar{N}, \mathcal{P}^{\Gamma}\right) \right|} \sum_{\sigma \in \Sigma(N)} MC^{v^L}_i (\sigma) \\
&= \frac{\left| \bar{N}(i) \right|! \prod_{j \in N, j \neq i} \left| \bar{N}(j) \right|!}{\left| \Sigma(N) \right| \prod_{j \in N} \left| \bar{N}(j) \right|!} \sum_{\sigma \in \Sigma(N)} MC^{v^L}_i (\sigma)
\end{aligned}$$

$$
\begin{aligned}
&= \left|\Sigma(N)\right|^{-1} \sum_{\sigma\in\Sigma(N)} MC_i^{v^L}(\sigma) \\
&\overset{(1)}{=} \mathrm{Sh}_i\left(N, v^L\right) \\
&\overset{(5)}{=} \mu_i(N, v, L),
\end{aligned}
$$

where the fifth equation is shown below, the sixth equation follows from $\bar{\sigma}|_N = \sigma$ for $\bar{\sigma} \in \Sigma_i\left(\bar{N}, \mathcal{P}^\Gamma, \sigma, \rho\right)$ by (8), and the ninth equation holds by (2) and (8).

The fifth equation can be seen as follows. Let $\bar{\sigma} \in \Sigma_i\left(\bar{N}, \mathcal{P}^\Gamma, \sigma, \rho\right)$ for $i \in N$, $\sigma \in \Sigma(N)$ and $\rho \in \Sigma\left(\bar{N}(i)\right)$. We then have

$$
\begin{aligned}
\sum_{(i,\lambda)\in\bar{N}(i)} MC_{(i,\lambda)}^{\bar{v}^{\bar{L}}}(\bar{\sigma}) &= \sum_{(i,\lambda)\in\bar{N}(i)} \bar{v}^{\bar{L}}\left(K_{(i,\lambda)}(\bar{\sigma})\right) - \bar{v}^{\bar{L}}\left(K_{(i,\lambda)}(\bar{\sigma}) \setminus \{(i,\lambda)\}\right) \\
&\overset{(8)}{=} \bar{v}^{\bar{L}}\left(\bigcup_{j\in K_i(\bar{\sigma}|_N)} \bar{N}(j)\right) - \bar{v}^{\bar{L}}\left(\bigcup_{j\in K_i(\bar{\sigma}|_N)\setminus\{i\}} \bar{N}(j)\right) \\
&\overset{(5)}{=} \sum_{\bar{S}\in\mathcal{C}\left(\bigcup_{j\in K_i(\bar{\sigma}|_N)} \bar{N}(j), \bar{L}|_{\bigcup_{j\in K_i(\bar{\sigma}|_N)}}\right)} \bar{v}\left(\bar{S}\right) \\
&\quad - \sum_{\bar{S}\in\mathcal{C}\left(\bigcup_{j\in K_i(\bar{\sigma}|_N)\setminus\{i\}} \bar{N}(j), \bar{L}|_{\bigcup_{j\in K_i(\bar{\sigma}|_N)\setminus\{i\}}}\right)} \bar{v}\left(\bar{S}\right) \\
&\overset{(6c)}{=} \sum_{\bar{S}\in\mathcal{C}\left(\bigcup_{j\in K_i(\bar{\sigma}|_N)} \bar{N}(j), \bar{L}|_{\bigcup_{j\in K_i(\bar{\sigma}|_N)}}\right)} v\left(N\left(\bar{S}\right)\right) \\
&\quad - \sum_{\bar{S}\in\mathcal{C}\left(\bigcup_{j\in K_i(\bar{\sigma}|_N)\setminus\{i\}} \bar{N}(j), \bar{L}|_{\bigcup_{j\in K_i(\bar{\sigma}|_N)\setminus\{i\}}}\right)} v\left(N\left(\bar{S}\right)\right)
\end{aligned}
$$

$$\stackrel{(6)}{=} \sum_{S\in\mathcal{C}\left(K_i(\bar{\sigma}|_N),L|_{K_i(\bar{\sigma}|_N)}\right)} v(S) - \sum_{S\in\mathcal{C}\left(K_i(\bar{\sigma}|_N)\setminus\{i\},L|_{K_i(\bar{\sigma}|_N)\setminus\{i\}}\right)} v(S)$$

$$\stackrel{(5)}{=} v^L\left(K_i\left(\bar{\sigma}|_N\right)\right) - v^L\left(K_i\left(\bar{\sigma}|_N\right)\setminus\{i\}\right)$$

$$= MC_i^{v^L}\left(\bar{\sigma}|_N\right).$$

This concludes the proof. □

References

Borm, P., Owen, G. and Tijs, S. (1992). On the position value for communication situations, *SIAM Journal on Discrete Mathematics* **5**: 305–320.

Casajus, A. (2007). The position value is the Myerson value, in a sense, *International Journal of Game Theory* **36**(1): 47–55.

Meessen, R. (1988). *Communication games*, Master's thesis, Department of Mathematics, University of Nijmegen, the Netherlands. (in Dutch).

Myerson, R. B. (1977). Graphs and cooperation in games, *Mathematics of Operations Research* **2**: 225–229.

Owen, G. (1977). Values of games with a priori unions, *in* R. Henn and O. Moeschlin (eds), *Essays in Mathematical Economics & Game Theory*, Springer, Berlin et al., pp. 76–88.

Shapley, L. S. (1953). A value for n-person games, *in* H. Kuhn and A. Tucker (eds), *Contributions to the Theory of Games*, Vol. II, Princeton University Press, Princeton, pp. 307–317.

Vázquez-Brage, M., García-Jurado, I. and Carreras, F. (1996). The Owen value applied to games with graph-restricted communication, *Games and Economic Behavior* **12**: 42–53.