Marginality is equivalent to coalitional strategic equivalence

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Abstract

Unlike otherwise claimed by Chun (1989, Games Econ Behav 1: 119–130), marginality (Young, 1985, Int J Game Theory 14: 65–72) and Chun’s coalitional strategic equivalence, both employed in characterizations of the Shapley value, are equivalent.

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1. Introduction

Characterizations of the Shapley value (1953) are abundant. One of the most beautiful such characterizations certainly is suggested by Young (1985). Besides efficiency and symmetry, which are quite standard, he employs the elegant marginality axiom.¹ In essence, marginality requires a player’s payoff only to depend on his own marginal contributions.

Later, Chun (1989) comes up with another characterization, which involves the coalitional strategic equivalence axiom. This axiom forces a player’s payoff not to be affected by adding unanimity games (possibly with some factor) in which he is a Null player. Chun’s characterization draws its appeal mainly from his allegation that coalitional strategic equivalence is weaker than marginality, which he demonstrates by some examples. However, this claim is wrong. In the following, we show that coalitional strategic equivalence and marginality are equivalent. Remarkably, this fact seems to be unrecognized in the literature (see e.g. Chun (1991, Remark 2); Khmelnitskaya (1999, p. 46)).

The plan of this note is as follows: Basic definitions and notation are given in second section. In the third section, we present our main result.

2. Basic definitions and notation

A TU game is a pair \((N,v)\) consisting of a non-empty and finite set of players \(N\) and the coalition function \(v \in V(N) := \{ f : 2^N \rightarrow \mathbb{R}, v(\emptyset) = 0 \}\). Subsets of \(N\) are called coalitions, and \(v(K)\) is called the worth of coalition \(K\). For \(T \in 2^N \setminus \{\emptyset\}\), the game \((N,u_T)\), \(u_T(K) = 1\) if \(T \subseteq K\) and \(u_T(K) = 0\) otherwise, is called a unanimity game. The sum \(v + w\) and product \(\lambda \cdot v\) with a scalar \(\lambda \in \mathbb{R}\), \(v,w \in V(N)\) are given by \((v + w)(K) = v(K) + w(K)\) and \((\lambda \cdot v)(K) = \lambda \cdot v(K)\) for all \(K \subseteq N\). It is well known that any \(v \in V(N)\) can be uniquely represented by unanimity games,

\[
v = \sum_{T \in 2^N \setminus \{\emptyset\}} \lambda_T(v) \cdot u_T, \quad \lambda_T(v) \in \mathbb{R},
\]

where the Harsanyi (1959) dividends \(\lambda_T(v)\) are given inductively by

\[
\lambda_{\{i\}}(v) = v(\{i\}) \quad \text{and} \quad \lambda_K(v) = v(K) - \sum_{\emptyset \neq K \subsetneq T} \lambda_K(v), \quad \emptyset \neq K \subseteq N.
\]

¹Originally, Young (1985, p. 71) refers to this axiom as an independence condition; following Chun (1989), we call it marginality.
The **marginal contribution** of \( i \in N \) to \( K \subseteq N \setminus \{i\} \) is given by \( MC_i^v(K) := v(K \cup \{i\}) - v(K) \). A **value** is an operator \( \varphi \) that assigns payoff vectors to all games, \( \varphi(N, v) \in \mathbb{R}^N \).

Finally, we consider the following axioms:

**Marginality, M.** If \( MC_i^v(K) = MC_i^w(K) \) for \( i \in N \) and all \( K \subseteq N \setminus \{i\} \) then \( \varphi_i(N, v) = \varphi_i(N, w) \).

**Coalitional strategic equivalence, CSE.** Let \( i \in N \) and \( \lambda \in \mathbb{R} \). If \( \emptyset \neq T \subseteq N \setminus \{i\} \) then \( \varphi_i(N, v) = \varphi_i(N, v + \lambda \cdot u_T) \).

3. **Coalitional strategic equivalence implies marginality**

In order to prepare the proof of the equivalence of \( \textbf{M} \) and \( \textbf{CSE} \), we first provide a simple lemma which establishes the relation between the hypothesis of \( \textbf{M} \) and the Harsanyi dividends.

**Lemma.** If \( i \in N \), then \( MC_i^v(K) = MC_i^w(K) \) for all \( K \subseteq N \setminus \{i\} \) iff \( \lambda_{K \cup \{i\}}(v) = \lambda_{K \cup \{i\}}(w) \) for all \( K \subseteq N \setminus \{i\} \).

**Proof.** By (1), \( v(K) = \sum_{T \subseteq K} \lambda_T(v) \) for \( K \subseteq N \). Thus, we have

\[
MC_i^v(K) = \sum_{T \subseteq K \cup \{i\}} \lambda_T(v) - \sum_{T \subseteq K} \lambda_T(v) = \sum_{T \subseteq K} \lambda_{T \cup \{i\}}(v)
\]

for all \( K \subseteq N \setminus \{i\} \). This already proves the \( \Leftarrow \)-part.

For \( K = \emptyset \), the \( \Rightarrow \)-claim is immediate. We proceed by induction on \( \vert K \vert \). Suppose the claim holds for all \( K \subseteq N \setminus \{i\} \), \( \vert K \vert \leq n \). Let \( K \subseteq N \setminus \{i\} \), \( \vert K \vert = n + 1 \) and \( MC_i^v(K) = MC_i^w(K) \) for all \( K \subseteq N \setminus \{i\} \) and \( i \in N \). We then have

\[
0 = MC_i^v(K) - MC_i^w(K) = \sum_{T \subseteq K} \lambda_{T \cup \{i\}}(v) - \sum_{T \subseteq K} \lambda_{T \cup \{i\}}(w) = \lambda_{K \cup \{i\}}(v) - \lambda_{K \cup \{i\}}(w),
\]

where the last equation follows from the induction hypothesis. This concludes the proof. \( \square \)

**Proposition.** \( \textbf{CSE} \) and \( \textbf{M} \) are equivalent.

\[\textit{Recently, van den Brink (2007, p. 770, footnote 3) demonstrates that coalitional strategic equivalence is equivalent to the following axiom: If } i \text{ is a Null player in } (N, w), \text{ then } \varphi_i(N, v) = \varphi_i(N, v + w).\]
Proof. By Chun (1989, Lemma 2), \( \textbf{M} \) implies \textbf{CSE}. Now, we turn to the other direction. Let \( i \in N, v, \) and \( w \) satisfy the hypothesis of \( \textbf{M} \), i.e. \( MC_i^u(K) = MC_i^w(K) \) for all \( K \subseteq N \setminus \{i\} \), and let \( \varphi \) obey \textbf{CSE}. Define

\[
q := v - \sum_{\emptyset \neq T \subseteq N \setminus \{i\}} \lambda_T(v) \cdot u_T = w - \sum_{\emptyset \neq T \subseteq N \setminus \{i\}} \lambda_T(w) \cdot u_T,
\]

where the second equation drops from the Lemma. Since \( v \) and \( q \) differ only by games of the form \( \lambda \cdot u_T, \lambda \in \mathbb{R}, T \subseteq N \setminus \{i\} \), successive application of \textbf{CSE} yields \( \varphi_i(N,v) = \varphi_i(N,w) \). Analogously, one derives \( \varphi_i(N,w) = \varphi_i(N,q) \). Hence, \textbf{CSE} implies \( \textbf{M} \).

\[\square\]

Remark. The Proposition entails that some of the allegations in Chun (1989, Section 4) are wrong. In particular, the value \( \phi^2 \) obeys not only \textbf{CSE}, but also \( \textbf{M} \). Moreover, some of the values \( \phi^4 \) neither meet \textbf{CSE} nor \( \textbf{M} \).

References


