Towards an evolutionary cooperative game theory

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Abstract

The idea behind evolutionary game theory is to interpret payoffs as fitness. Non-cooperative game theory has seen many interesting results and applications. The aim of this paper is to introduce a replicator-dynamics version for cooperative game theory. We generate payoffs, or fitness values, by way of the Mertens (1980) value which is a close cousin of the continuous Shapley value introduced by Aumann & Shapley (1974). We already present some preliminary results for the resulting replicator dynamics (this part is work in progress). We close with some (well-founded) conjectures on the evolutionary dynamics for particular TU games.

Keywords: Mertens value, replicator dynamics, asymptotic stability, apex game

JEL classification: C71

1. Introduction

Evolutionary models of various forms have been part and parcel of economics for a long time (see, for example, the articles collected by Witt 1993). A specific class of models has been developed within game theory. In usual parlance, evolutionary game theory (see, for example, Weibull (1995) or Samuelson (1997)) means evolutionary theory applied to non-cooperative games. The aim of this paper is to develop an evolutionary cooperative game theory where we concentrate on the transferable-utility case. Apparently, ideas in this direction have been around for some time. Nasar (2002, p. xxiv) reports that John Nash, picking up his old interest in game theory, "received a grant from the National Science Foundation to develop a new 'evolutionary' solution concept for cooperative games".

Cooperative game theory rests on two pillars. First, the economic (or political or sociological ...) situation is described by a player set N and a coalition function $v: 2^N \to \emptyset$. Subsets of N are called coalitions, with N being the grand coalition. For every coalition $K \subseteq N$, $v(K)$ is its worth that stands for the possibilities open to that coalition. For $T \in 2^N \backslash \{\emptyset\}$, the game u_T is called the unanimity game; it is defined by $u_T(K) = 1$ if $T \subseteq K$ and $u_T(K) = 0$ otherwise. The players from T are often called the productive players while the other players are unproductive. As a second example, consider the apex game h for $N = \{1, ..., 4\}$. It is defined by

$$h(K) = \begin{cases} 1, & 1 \in K \text{ and } K \backslash \{1\} \neq \emptyset \\ 1, & K = N \backslash \{1\} \\ 0, & \text{otherwise} \end{cases}$$

Thus, the apex player 1 needs one additional player to produce the worth 1. This worth can also be created by the coalition of the three less important players 2, 3, and 4.

Cooperative game theory's second pillar are solution concepts, the most famous being the core and the Shapley's (1953) value. From the point of view of applicability, the Shapley value has the advantage of producing unique payoffs for the n players. Thus, the coalition function is the input into a solution concept and the payoff vector the output. For the above apex game, the Shapley payoff vector is

$$\text{Sh}(N, v) = \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right).$$

Aumann & Myerson (1988) interpret the apex game as a weighted voting game and try to predict coalition formation. They present a hybrid noncooperative-cooperative model which, following Brandenburger & Stuart (2007) (who use the core rather than the Shapley value or the weighted XP Shapley value), can also be called a biform games. In that model, players (political parties) sequentially decide on forming links with other players. Depending on the links, the players obtain the Myerson (1977) value which is an adaptation of the Shapley value for networks. Aumann & Myerson (1988) find that backward induction applied to the link game leads to the coalition $\{2,3,4\}$ where the three little players join to share the worth of 1.

In order to address problems of coalition formation (and for other purposes as well), we take another approach by developping evolutionary cooperative game theory. Imbedding games like the apex game into an evolutionary setting, we interpret the payoffs as fitness. A player's success feeds into his proliferation. In order to model reproductive differences between players, we distinguish between players (like the four players in the apex game) and agents who take up the roles (or types) of these n players. If a role is particularly fruitful (in producing relatively high payoffs for the agents assuming that role), the relative number of these agents increases.

While the number of players is a natural number, we deal with a continuum of agents for each player. Therefore, we need an extended coalition function that is capable of dealing with non-integer players (agents). The Lovasz extension $\bar{v}$ or the multi-linear (Owen) extension v^{MLE} are suitable candidates. We prefer the Lovasz extension to the multi-linear extension. If players (or agents) work together (in the framework of a unanimity or an apex game) and if the size of the agents is below 1, the multilinear extension has a probabilistic interpretation (as noted by Owen 1972, p. 64) – the players work together only if their time schedules happen to coincide. For example, two productive players in the unanimity game $u_{\{1,2\}}$ with $s=\left(\frac{1}{2},\frac{1}{3}\right)$ can produce $\frac{1}{2}\cdot\frac{1}{3}$, only. It seems to us that (by appropriate coordination), the two agents should be able to produce the minimum of these two figures, $\frac{1}{3}$,. which is exactly what the Lovasz extension does. Also, consider $s=(2,3)$. The multi-linear extension yields $2\cdot 3=6$ whereas the minimum extension leads to 2. Also, an extension's worth may turn out to be negative even if the underlying coalition function itself is positive. In fact, we find $h^{MLE}(2,1,3,4)=-10$. The use of the Lovasz extension has important repercussions for our model. In unanimity games, the productive players with minimal agent sets get all the payoff.

Extensions of coalition functions cannot be an input for the (standard) Shapley value. While the Lovasz extension $\bar{v}$ is continuous, it is not differentiable in general. Hence, the diagonal formula introduced by Aumann & Shapley (1974) is not applicable. Fortunately, however, we can use the Mertens (1980) value as Haimanko (2001) has shown for a formally similar problem (of cost division). The Mertens value thus yields the payoff information understood as fitness and we can then define the replicator dynamics. Whenever an agent receives an above-average payoff, the population share of his player will increase – a standard result

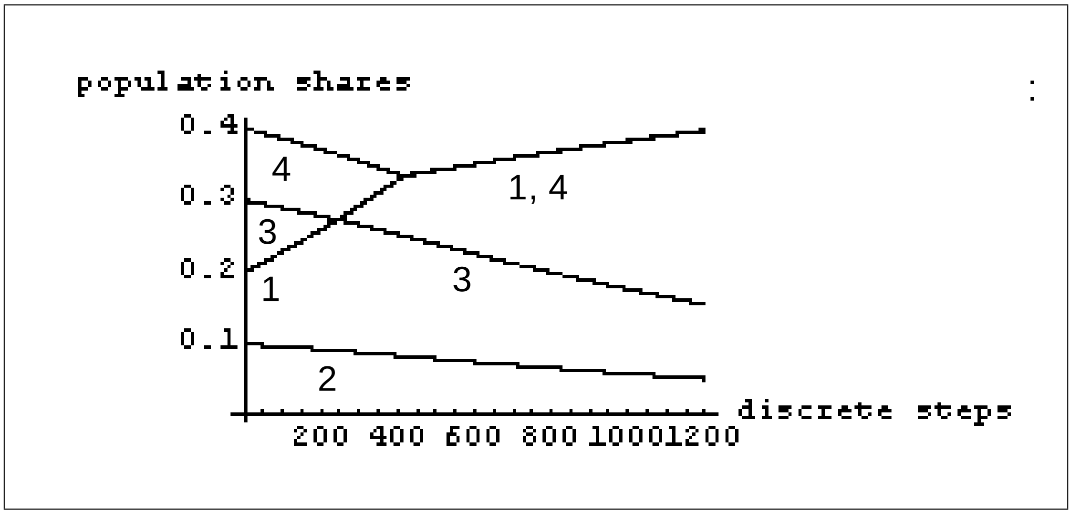

Figure 1.1: The apex player teams up with player 4

in replicator-dynamics models (for example Weibull 1995, chapter 3). We strive to derive the differential equation of the resulting replicator dynamics in this paper.

The reader is invited to consider two figures. They report the results of a discrete-step analogue of the replicator dynamics' differential equation. The initial population share vector $x(0) = \left(\frac{2}{10}, \frac{1}{10}, \frac{3}{10}, \frac{4}{10}\right)$ is used in fig. 1.1. The apex player's initial share is at least as high as the share of any of the weak players. In that case, the apex player teams up with the weak player who has the largest initial share. In the beginning, only player 1's agent set grows. As soon as the sizes of player 1's agent set and player 4's agent set equal, both agent sets grow while the agent sets of players 2 and 3 tend towards zero. Thus, the asymptotically stable population share vector is $\hat{x} = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$.

Fig. 1.2 starts with the population share vector $\left(\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}\right)$, i.e., the apex player's initial share is lower than the shares of all the weak players. In this case, the apex player's share tends to zero and the asymptotically stable population share vector is $\hat{x} = \left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. This second example closely resembles the results found by Aumann & Myerson (1988).

In terms of interpretation, evolutionary noncooperative game theory (ENGT) differs from evolutionary cooperative game theory (ECGT). ENGT builds on the idea that two players are drawn at random from a large population. They are programmed to play a certain (mixed) strategy and the strategy that does better

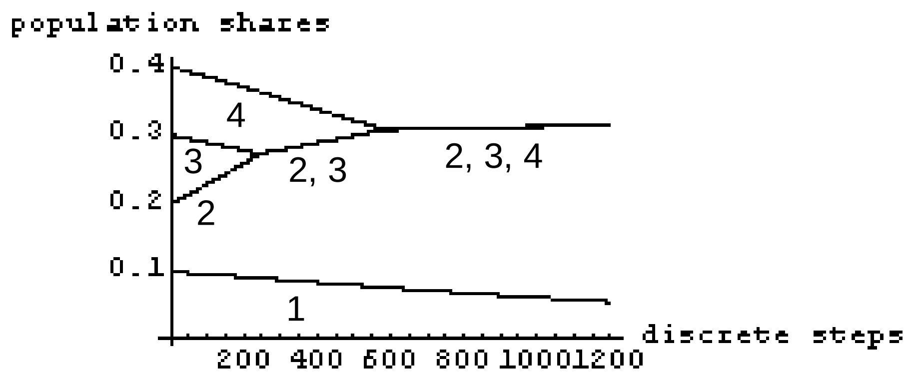

Figure 1.2: The three unimportant players trump the apex player

than other strategies grows faster. The most basic model of ENGT builds on (a) "pairwise contests" and (b) a monomorphic population playing a symmetric game. Of course, more advanced ENGT models also deal with polymorphic playing-the-field situations.

In contrast, the "simple" ECGT as presented here naturally belongs to (a) the "playing the field" and (b) the polymorphic variety. (a) just results from the way the Mertens value is calculated and interpreted – an agent's payoff depends on the set-up of the economy as a whole. (b) is the natural outflow from different players' roles.

We suggest to reserve the term ECGT for the non-atomic (or at least many-agents) setup. It is the agents whose shares change. In contrast, one might envision a model where the players themselves grow or shrink. A suitable example is provided by firms. Depending on their profits they will grow in an organic fashion (rather than grow by mergers and acquisitions). This is not the kind of model we have in mind in this paper.

Our work is related to the concept of "dynamic cooperative games" introduced by Filar & Petrosjan (2000). Their idea is to define a sequence of games (in discrete or in continuous time) so that one TU game is determined by the previous one and by the payoffs achieved under some solution concept. The players (no agents involved) in that paper obtain the sum of payoffs for this sequence of coalition functions. The authors deal with the problem of whether these payoffs obey some consistency criterion.

When discussion the apex game, we argue that ECGT migth be considered a contribution to the vast field of coalition formation. Alternatively, while a major application of, and motivation for, ENGT is equilibrium selection (see the titel of the book by Samuelson 1997), ECGT focuses on the evolutionary pressure against players (rather than strategy combinations). In particular, we derive the following results: 1. Dominated players (with lower marginal contributions) may survive in the long run. 2. For simple games, asymptotically stable population share vectors involve minimal winning coalitions.

The paper is organized as follows. In the following section, we provide the basics of TU games together with a definition of the Shapley value. Section 3 introduces vector measure games for Lovasz extensions. We are then set to derive payoffs by way of the Mertens value in section 4. We already present some preliminary results for the resulting replicator dynamics (this part is work in progress) in section 5. We then comment on some stability results in the section 6 before concluding the paper with section 7.

2. Basic definitions and notation

2.1. Payoff vectors, population-size vectors

For a non-empty and finite (player) set N, let

$$\begin{aligned}
\mathbb{R}_{+}^{N} &:= \left\{x \in \mathbb{R}^{N} | \forall i \in N : x_i \geq 0\right\}, \\
\mathbb{R}_{++}^{N} &:= \left\{x \in \mathbb{R}^{N} | \forall i \in N : x_i > 0\right\}, \\
\Delta_{+}^{N} &:= \left\{x \in \mathbb{R}_{+}^{N} | \textstyle\sum_{i \in N} x_i = 1\right\}, \\
\Delta_{++}^{N} &:= \Delta_{+}^{N} \cap \mathbb{R}_{++}^{N}.
\end{aligned}$$

2.2. Finite TU games

Let $\mathcal{U}$ be a sufficiently large infinite set, the universe of players; $\mathbb{N}(\mathcal{U})$ denotes the set of non-empty and finite set of subsets of $\mathcal{U}$. A **(TU) game** on $\mathcal{U}$ is a pair (N, v) consisting of a set of players $N \in \mathbb{N}(\mathcal{U})$ and a **coalition function** $v \in \mathbb{V}(N) := \left\{f : 2^N \to \mathbb{R} | f(\emptyset) = 0\right\}$. Subsets of N are called **coalitions**, and $v(K)$ is called the worth of coalition K.

For $v, w \in \mathbb{V}(N)$, $\alpha \in \mathbb{R}$, the coalition functions $v + w \in \mathbb{V}(N)$ and $\alpha \cdot v \in \mathbb{V}(N)$ are given by $(v + w)(K) = v(K) + w(K)$ and $(\alpha \cdot v)(K) = \alpha \cdot v(K)$ for all

$K \subseteq N$. For $K \subseteq N$ and $v \in \mathbb{V}(N)$, $v|_K \in \mathbb{V}(K)$ denotes the restriction of v to 2^K. The **null game** on N is denoted $(N, \mathbf{0})$, $\mathbf{0} \in \mathbb{V}(N)$, where $\mathbf{0}(K) = 0$ for all $K \subseteq N$. For $T \in 2^N \setminus \{\emptyset\}$, the game (N, u_T), $u_T(K) = 1$ if $T \subseteq K$ and $u_T(K) = 0$ otherwise, is called a **unanimity game**; the game (N, e_T), $e_T(K) = 1$ if $T = K$ and $e_T(K) = 0$ otherwise, is called a **standard game**. A game (N, v) is called **simple** iff $v(K) \in \{0, 1\}$ for all $K \subseteq N$; it is called **superadditive** iff $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$, $S \cap T = \emptyset$. Let $\mathbb{V}^{\mathrm{sa}}(N)$ and $\mathbb{V}^{\mathrm{si}}(N)$ denote the sets of superadditive and of simple coalition functions on N, respectively. Any $v \in \mathbb{V}(N)$ can be uniquely represented by unanimity games,

$$v = \sum_{T \subseteq N : T \neq \emptyset} \lambda_T(v) \cdot u_T, \qquad \lambda_T(v) := \sum_{S \subseteq T : S \neq \emptyset} (-1)^{|T|-|S|} \cdot v(S). \tag{2.1}$$

Player $i \in N$ is called a **dummy player** in (N, v) iff $v(K \cup \{i\}) - v(K) = v(\{i\})$ for all $K \subseteq N \setminus \{i\}$; if in addition $v(\{i\}) = 0$, then i is called a **null player**; players $i, j \in N$ are called **symmetric** in (N, v) if $v(K \cup \{i\}) = v(K \cup \{j\})$ for all $K \subseteq N \setminus \{i, j\}$.

A **value** on N is an operator φ that assigns a payoff vector $\varphi(N, v) \in \mathbb{R}^N$ to any game (N, v). An **order** of a set N is a bijection $\rho : N \to \{1, \ldots, |N|\}$ with the interpretation that i is the $\rho(i)$th player in ρ. The set of these orders is denoted by $R(N)$. The set of players weakly preceding i in ρ is denoted by $K_i(\rho) = \{j \in N : \rho(j) \leq \rho(i)\}$. For $i \notin K$, the **marginal contribution** of i at K is defined by $MC_i^v(K) := v(K \cup \{i\}) - v(K)$. The **marginal contribution** of i under ρ is denoted $MC_i^v(\rho) := MC_i^v(K_i(\rho) \setminus \{i\})$. The **Shapley value** on $\mathbb{V}(N)$, Sh, is given by

$$\mathrm{Sh}_i(N, v) := \frac{1}{|R(N)|} \sum_{\rho \in R(N)} MC_i^v(\rho), \qquad i \in N, v \in \mathbb{V}(N). \tag{2.2}$$

3. Vector measure games

3.1. Definition

Let $\mathcal{I}$ be a set and $\mathcal{C}$ a subset of $2^{\mathcal{I}}$ such that $(\mathcal{I}, \mathcal{C})$ is isomorphic to $([0, 1], \mathcal{B})$, where $\mathcal{B}$ stands for the Borel subsets of $[0, 1]$. A game on $(\mathcal{I}, \mathcal{C})$ is a mapping $v : \mathcal{C} \to \mathbb{R}$ such that $v(\emptyset) = 0$. A game is **finitely additive** if $v(S \cup T) = v(S) \cup v(T)$ for all $S, T \in \mathcal{C}$, $S \cap T = \emptyset$. A **value** φ on $(\mathcal{I}, \mathcal{C})$ is linear operator that assigns to any game v a finitely additive game φv and that is symmetric, positive, and efficient (see Neyman 2002, Section 3).

game	coalition	solution conc.
TU game (N,v)	$K \in 2^N$	Shapley value
vector measure game	$C \in \mathcal{C}$	Diagonal formula: a) Aumann/Shapley b) Mertens
Lovasz extension $(N,\overline{v})$	$s \in R_+^N$	
derived vector measure game $\widetilde{v}^s = \left(N, \left(\mu_i^s\right)_{i\in N}, \overline{v}\right)$	$C \in \mathcal{C}$	Mertens value

Figure 3.1: From a TU game to a vector measure game

A game v on $(\mathcal{I}, \mathcal{C})$ is called a **vector measure game** if there is a triple

$$\left(N, (\mu_i)_{i\in N}, f\right),$$

where N is a non-empty and finite set N, μ_i, $i \in N$ is non-atomic on $(\mathcal{I}, \mathcal{C})$, and f is a function $f : \mathbb{R}_+^N \to \mathbb{R}$ such that

$$v := f \circ \mu,$$

where μ is the vector measure $\mu : \mathcal{C} \to \mathbb{R}_+^N$ given by

$$\mu(C) = (\mu_i(C))_{i\in N}, \qquad C \in \mathcal{C}.$$

Abusing notation, we the write $v = \left(N, (\mu_i)_{i\in N}, f\right)$.

Consider fig. 3.1. So far, we have presented TU games and vector measure games in general. The next two subsections are devoted to the Lovasz extension and the derived vector measure game while section 4 deals with the solution concepts.

3.2. The Lovasz extension

The Lovasz (1983) extension $(N, \bar{v})$, $\bar{v} : \mathbb{R}_+^N \to \mathbb{R}$ of a TU game $(N, \bar{v})$ is given by

$$\bar{v}(s) := \sum_{T \subseteq N, T \neq \emptyset} \lambda_T(v) \cdot \min_T(s), \qquad s \in \mathbb{R}_+^N, \tag{3.1}$$

where

$$\min_K : \mathbb{R}^N \to \mathbb{R}, \qquad s \mapsto \min_K(s) := \min_{i \in K} s_i, \qquad K \subseteq N, K \neq \emptyset. \tag{3.2}$$

For unanimity games this implies

$$\bar{u}_T(s) = \min_T(s), \qquad T \subseteq N, T \neq \emptyset, s \in \mathbb{R}_+^N. \tag{3.3}$$

By (3.1) and (3.2), the Lovasz extension is linear in the coalition function, $\overline{\alpha v + \beta w} = \alpha \cdot \bar{v} + \beta \cdot \bar{w}$ for all $v, w \in \mathbb{V}(N)$, and $\alpha, \beta \in \mathbb{R}$, and non-negatively homogenous in s, $\bar{v}(\alpha \cdot s) = \alpha \cdot \bar{v}(s)$ for all v and $\alpha \in \mathbb{R}_+$ (see Lovasz 1983, Algaba, Bilbao, Fernandez & Jimenez 2004).

Note that while the min-operator is continuous, it is not partially differentiable. Of course, this peculiarity is passed on to $\bar{v}$.

3.3. Derived vector measure game

From (N, v), $v \in \mathbb{V}(N)$ and $s \in \mathbb{R}_+^N$, we construct a vector measure game $\tilde{v}^s = \left(N, (\mu_i^s)_{i \in N}, \bar{v}\right)$ on $(\mathcal{I}, \mathcal{C})$ (isomorphic to $([0,1], \mathcal{B})$) as follows: There is a collection $(A_i)_{i \in N}$, $A_i \in \mathcal{C}$ with the following properties:

D1 μ_i^s, $i \in N$ is a measure on $(\mathcal{I}, \mathcal{C})$.

D2 For all $i, j \in N$, $i \neq j$, we have $A_i \cap A_j = \emptyset$.

D3 $\mu_i^s(C) = \mu_i^s(A_i \cap C)$, $i \in N$, $C \in \mathcal{C}$.

D4 $\mu_i^s(A_i) = s_i$, $i \in N$.

Remark 1. *The intendend interpretation is the following: The set $\mathcal{I}$ contains the agents of the players in N. Since we do not require the collection $(A_i)_{i \in N}$ to be exhaustive, $\mathcal{I}$ may contain additional agents, which are, however, completely unproductive. The set $\mathcal{C}$ contains all coalitions under consideration. For all $i \in N$, the coalition A_i contains all agents of player i and no agents of the players $j \in N \backslash \{i\}$. The measure μ_i^s, $i \in N$ assigns to any coalition $C \in \mathcal{C}$ the number of* *agents of player i contained in it. Overall, there are $\mu_i^s(\mathcal{I}) = \mu_i^s(A_i) = s_i$ agents of player i which only inhabit A_i. The vector measure μ assigns to any $C \in \mathcal{C}$ the vector $\mu^s(C)$ of "population sizes" of the players from N in C. Finally, the Lovasz extension $\bar{v}$ determines the worth generated by any $C \in \mathcal{C}$ based on these "population sizes".*

Remark 2. *It is easy to see that such a vector measure game $\tilde{v}^s = \left(N, (\mu_i^s)_{i\in N}, \bar{v}\right)$ on some $(\mathcal{I}, \mathcal{C})$ always exists. Let $(\mathcal{I}, \mathcal{C}) = ([0,1], \mathcal{B})$. Consider a injection $\xi : N \to \{1, \ldots |N|\}$ and let $A_i := \left(\frac{\xi(i)-1}{|N|}, \frac{\xi(i)}{|N|}\right)$ and μ_i^s be given by the density function $\delta_i^s : [0,1] \to \mathbb{R}$,*

$$\delta_i^s(x) := \begin{cases} |N| \cdot s_i, & x \in A_i, \\ 0, & x \in \mathcal{I} \backslash A_i, \end{cases} \qquad x \in [0,1], i \in N,$$

and the Lebesgue measure on $[0,1]$.

Remark 3. *Given (N, v) and $s \in \mathbb{R}_+^N$, there is a lot of arbitrariness in choosing $(\mathcal{I}, \mathcal{C})$ and $(A_i, \mu_i^s)_{i\in N}$ as above. Ultimately, we are interested in the payoffs of the coalitions A_i, $i \in N$ only. Under the value applied, as we will see later on, these payoffs are not sensitive to the unspecified details.*

4. Payoffs for the vector measure game

Now, we determine the payoffs $\varphi(A_i)$, $i \in N$ for the vector measure games $\tilde{v}^s = \left(N, (\mu_i^s)_{i\in N}, \bar{v}\right)$, $s \in \mathbb{R}_+^N$, on $(\mathcal{I}, \mathcal{C})$.

Note that for vector measure games $v = \left(N, (\mu_i)_{i\in N}, f\right)$ on $(\mathcal{I}, \mathcal{C})$, where f is continuously differentiable and $\mu_i(\mathcal{I}) > 0$, $i \in N$, there is a convenient way to determine the value, the **diagonal formula** by Aumann & Shapley (1974, Theorem B), also see Neyman (2002, pp. 2141). Alas, since the Lovasz extension $\bar{v}$ is not differentiable in general, we build on this approach.

Fortunately, however, the games $\tilde{v}^s$, $s \in \mathbb{R}_+^N$ belong to a class of games on $(\mathcal{I}, \mathcal{C})$ on which the Mertens value (Mertens 1980) is the unique value (Neyman, 2002, Section 8; Haimanko, 2001). In particular, Haimanko deals with vector measure games $v = \left(N, (\mu_i)_{i\in N}, f\right)$, with following properties:

H1 μ_i, $i \in N$ is a probability measure.

H2 The probability measures μ_i, $i \in N$ are mutually singular, i.e., they have pairwise different carriers.

H3 The function f is continuous and piecewise linear.

It is easy to see that $\tilde{v}^s = \left(N, (\mu_i^s)_{i\in N}, \bar{v}\right)$ can be written to match this specification, i.e.,

$$\tilde{v}^s = \left(N, (\mu_i^s)_{i\in N}, \bar{v}\right) = \left(N, (\hat{\mu}_i^s)_{i\in N}, \bar{v}^s\right). \tag{4.1}$$

For $i \in N$, $s_i > 0$, the probability measure μ_i is given by

$$\hat{\mu}_i^s(C) = \frac{\mu_i^s(C)}{s_i}; \tag{4.2}$$

for $i \in N$, $s_i = 0$, choose any probability measure $\hat{\mu}_i^s$ such that $\hat{\mu}_i^s(A_i) = 1$ and $\hat{\mu}_i^s(C) = \hat{\mu}_i^s(A_i \cap C)$, $i \in N$, $C \in \mathcal{C}$. Further, let

$$h^s : \mathbb{R}_+^N \to \mathbb{R}_+^N, \qquad (x_i)_{i\in N} \mapsto h^s(x) = (s_i \cdot x_i)_{i\in N}, \qquad x \in \mathbb{R}_+^N \tag{4.3}$$

and set

$$\bar{v}^s = \bar{v} \circ h^s. \tag{4.4}$$

By (4.2), **D1**, and **D4**, $\left(N, (\mu_i)_{i\in N}, \bar{v}^s\right)$ meets **H1**. By (4.2), **D1**, and **D4**, $\left(N, (\mu_i)_{i\in N}, \bar{v}^s\right)$ obeys **H2**. To see, **H3**, set

$$\begin{aligned} \mathbb{R}_+^N(\rho) &: = \mathbb{R}_+^N \cap \mathbb{R}^N(\rho), \\ \mathbb{R}^N(\rho) &: = \left\{x \in \mathbb{R}^N \middle| \forall i, j \in N : \rho(i) < \rho(j) \iff x_i > x_j\right\}, \qquad \rho \in R(N). \end{aligned} \tag{4.5}$$

By (3.1), (3.2), and (4.5), $\bar{v}$ is continuous on $\mathbb{R}_+^N$ and linear on the closure of $\mathbb{R}_+^N(\rho)$, $\rho \in R(N)$. By (4.3) and (4.4), these properties are pased on from $\bar{v}$ to $\bar{v}^s$.

$$\begin{aligned} \mathbb{R}_{++}^N(\rho) &: = \mathbb{R}_{++}^N \cap \mathbb{R}_+^N(\rho) \\ \gneqq_+^N(\rho) &: = \gneqq_+^N \cap \mathbb{R}_+^N(\rho) \\ \gneqq_{++}^N(\rho) &: = \gneqq_+^N \cap \mathbb{R}_{++}^N(\rho) \\ \mathbb{R}_+^N(R) &: = \bigcup_{\rho\in R(N)} \mathbb{R}_+^N(\rho) \\ s &\in \mathbb{R}_+^N(R) \end{aligned}$$

The Mertens value φ_M on the class of vector measure games $\left(N, (\mu_i)_{i\in N}, f\right)$ obeying **H1–H3** is given by the following diagonal formula (Neyman 2002, pp. 2150)[1]:

$$\varphi_M (f \circ \mu)(C) = E_Y \int_0^1 \partial f (t \cdot \mathbf{1}_N; Y; \mu(C))\, dt, \qquad C \in \mathcal{C}, \tag{4.6}$$

where $\mathbf{1}_N \in \mathbb{R}^N$, $(\mathbf{1}_N)_i = 1$, $i \in N$, $Y = (Y_i)_{i\in N}$ is a vector of independent random varibles, each with the standard Cauchy distribution and

$$\partial f (x, y, z) := \lim_{\varepsilon\downarrow 0} \frac{f_{y+\varepsilon z}(x) - f_y(x)}{\varepsilon \cdot \|z\|}, \qquad x \in (0,1)^N,\ y, z \in \mathbb{R}^N, \tag{4.7}$$

where $f_y(x)$, $x \in (0,1)^N$, $y \in \mathbb{R}^N$ is the directional derivative of f at x in the direction of y. Note that for each fixed x, $\partial f(x,y,z)$ is the directional derivative of the function $y \mapsto f_y(x)$ at y in the direction of $z \in \mathbb{R}^N$.[2]

Recall that the density d of a random variable Y on $\mathbb{R}$ with the standard Cauchy distribution is

$$d(y) = \frac{1}{\pi(1+y^2)}, \qquad x \in \mathbb{R}.$$

Hence, the density d of a vector $Y = (Y_i)_{i\in N}$ of independent random variables, each with standard Cauchy distribution, on $\mathbb{R}^N$ is

$$d(y) = \prod_{i\in N} \frac{1}{\pi(1+y_i^2)}, \qquad y \in \mathbb{R}^N. \tag{4.8}$$

Since both the Lovasz extention and Mertens value are linear in the coalition function, respectively, we have

$$\begin{aligned} \varphi_M(\bar{v} \circ \mu^s)(A_i) &= \varphi_M(\bar{v}^s \circ \hat{\mu}^s)(A_i) \\ &= \sum_{T\subseteq N, T\neq\emptyset} \lambda_T(v) \cdot \varphi_M(\bar{u}_T^s \circ \hat{\mu}^s)(A_i) \end{aligned} \tag{4.9}$$

for all $i \in N$ and $v \in \mathbb{V}(N)$.

[1] Note that there is a misprint on Neyman (2002, p. 2150)—the obvoius expectation operator E_Y is missing.

[2] Since z is not normalized to $\|z\| = 1$, the nominator of the above formula should be $\varepsilon \cdot \|z\|$ and not just z as on Neyman (2002, p. 2150).

Fix $i \in N$ and $T \subseteq N, T \neq \emptyset$. In order to determine $\varphi_M\left(\bar{u}_T^s \circ \hat{\mu}^s\right)(A_i)$, we first calculate the expression

$$\partial \bar{u}_T^s\left(t \cdot \mathbf{1}_N; y; \hat{\mu}^s(A_i)\right), \qquad t \in [0,1], y \in \mathbb{R}^N \setminus \{\mathbf{0}_N\}.$$

By **D4** and (4.2), we have $\hat{\mu}^s(A_i) = \mathbf{1}_i \in \mathbb{R}^N$, $(\mathbf{1}_i)_i = 1$, $(\mathbf{1}_i)_j = 0$ for $j \in N \setminus \{j\}$. Further by (4.7),

$$\begin{aligned}
&\partial \bar{u}_T^s\left(t\mathbf{1}_N; y; \hat{\mu}^s(A_i)\right) \\
&\quad = \lim_{\varepsilon \downarrow 0} \frac{\left(\bar{u}_T^s\right)_{y+\varepsilon \cdot \mathbf{1}_i}(t \cdot \mathbf{1}_N) - \left(\bar{u}_T^s\right)_y (t \cdot \mathbf{1}_N)}{\varepsilon \cdot \|\mathbf{1}_i\|} \\
&\quad = \lim_{\varepsilon \downarrow 0} \frac{\lim_{\alpha \downarrow 0}\left[\dfrac{\bar{u}_T^s\left(t \cdot \mathbf{1}_N + \alpha \cdot (y + \varepsilon \cdot \mathbf{1}_i)\right) - \bar{u}_T^s(t \cdot \mathbf{1}_N)}{\alpha \cdot \|y + \varepsilon \cdot \mathbf{1}_i\|} - \ldots\right.}{\varepsilon} \\
&\qquad\qquad \frac{\left.\ldots \dfrac{\bar{u}_T^s(t \cdot \mathbf{1}_N + \alpha \cdot y) - \bar{u}_T^s(t \cdot \mathbf{1}_N)}{\alpha \cdot \|y\|}\right]}{}, \qquad (4.10)
\end{aligned}$$

where $\|\cdot\|$ denotes the Euklidean norm operator.

For $K \subseteq N$, $K \neq \emptyset$, set $\operatorname{argmin}_K(x) := \{j \in K | x_j = \min_K(x)\}$, $x \in \mathbb{R}^N$. We distinguish some cases.

Case (i): We consider $i \in N \setminus \operatorname{argmin}_T(s)$. By (3.2), (3.3), and (4.4), we have

$$\bar{u}_T^s(t \cdot \mathbf{1}_N) = t \cdot \min_T(s). \qquad (4.11)$$

Further, for sufficiently small ε and α,

$$\bar{u}_T^s\left(t \cdot \mathbf{1}_N + \alpha \cdot (y + \varepsilon \cdot \mathbf{1}_i)\right) = \bar{u}_T^s(t \cdot \mathbf{1}_N + \alpha \cdot y) = \min_T(s) \cdot \left(t + \alpha \cdot \min_{\operatorname{argmin}_T(s)}(y)\right).$$

Hence, (4.10) becomes

$$\begin{aligned}
\partial \bar{u}_T^s\left(t\mathbf{1}_N; y; \hat{\mu}^s(A_i)\right) &= \min_T(s) \cdot \min_{\operatorname{argmin}_T(s)}(y) \cdot \lim_{\varepsilon \downarrow 0} \frac{\left[\dfrac{1}{\|y + \varepsilon \cdot \mathbf{1}_i\|} - \dfrac{1}{\|y\|}\right]}{\varepsilon} \\
&= -\frac{y_i}{\|y\| \, \|y\|^2} \cdot \min_T(s) \cdot \min_{\operatorname{argmin}_T(s)}(y). \qquad (4.12)
\end{aligned}$$

For $y \in \mathbb{R}^N$, let $y^{(-i)} \in \mathbb{R}^N$ be given by $y_i^{(-i)} = -y_i$ and $y_j^{(-i)} = -y_j$ for $j \in N \setminus \{i\}$. By (4.12), we have

$$\partial \bar{u}_T^s\left(t\mathbf{1}_N; y^{(-i)}; \hat{\mu}^s(A_i)\right) = -\partial \bar{u}_T^s\left(t\mathbf{1}_N; y; \hat{\mu}^s(A_i)\right). \qquad (4.13)$$

By (4.8), $d\left(y^{(-i)}\right) = d(y)$, $y \in \mathbb{R}^N$. Hence, (4.13), (4.6), and (4.1) already entail

$$\varphi_M(\bar{u}_T \circ \mu^s)(A_i) = 0. \tag{4.14}$$

Case (ii): $i \in \operatorname{argmin}_T(s)$. Since Y is non-atomic, we are allowed to restrict attention to

$$y \in \mathbb{R}^N(R) = \bigcup_{\rho \in R(N)} \mathbb{R}^N(\rho).$$

Case (iia): $i \neq \operatorname{argmin}_{\operatorname{argmin}_T(s)}(y)$.

The arguments of *Case (i)* apply up to (4.12), i.e.,

$$\partial \bar{u}_T^s(t\mathbf{1}_N; y; \hat{\mu}^s(A_i)) = -\frac{y_i}{\|y\|\,\|y\|^2} \cdot \min_T(s) \cdot \min_{\operatorname{argmin}_T(s)}(y) \tag{4.15}$$

Case (iib): $i = \operatorname{argmin}_{\operatorname{argmin}_T(s)}(y)$. For sufficiently small ε and α,

$$\begin{aligned}\bar{u}_T^s(t \cdot \mathbf{1}_N + \alpha \cdot (y + \varepsilon \cdot \mathbf{1}_i)) &= \min_T(s) \cdot \left(t + \alpha \cdot \left(\min_{\operatorname{argmin}_T(s)}(y) + \varepsilon\right)\right) \\ \bar{u}_T^s(t \cdot \mathbf{1}_N + \alpha \cdot y) &= \min_T(s) \cdot \left(t + \alpha \cdot \min_{\operatorname{argmin}_T(s)}(y)\right).\end{aligned}$$

Hence together with (4.11), (4.10) becomes

$$\begin{aligned}\partial \bar{u}_T^s(t\mathbf{1}_N; y; \hat{\mu}^s(A_i)) &= \min_T(s) \cdot \lim_{\varepsilon \downarrow 0} \frac{\frac{y_i + \varepsilon}{\|y + \varepsilon \cdot \mathbf{1}_i\|} - \frac{y_i}{\|y\|}}{\varepsilon} \\ &= \min_T(s) \cdot \frac{1}{\|y\|} \frac{\|y\|^2 - y_i^2}{\|y\|^2}.\end{aligned} \tag{4.16}$$

For $y \in \mathbb{R}^N$, $i, j \in \operatorname{argmin}_T(s)$, $i \neq j$, let $y^{(i \leftrightarrow j)} \in \mathbb{R}^N$ be given by $y_i^{(i \leftrightarrow j)} = y_j$, $y_j^{(i \leftrightarrow j)} = y_i$ and $y_j^{(i \leftrightarrow j)} = -_j$ for $j \in N \setminus \{i, j\}$. By (4.15) and (4.16), we have

$$\partial \bar{u}_T^s\left(t\mathbf{1}_N; y^{(i \leftrightarrow j)}; \hat{\mu}^s(A_i)\right) = \partial \bar{u}_T^s(t\mathbf{1}_N; y; \hat{\mu}^s(A_j)). \tag{4.17}$$

Further by (4.8), $d\left(y^{(i \leftrightarrow j)}\right) = d(y)$, $y \in \mathbb{R}^N$. Hence, (4.17), (4.6), and (4.1) already entail

$$\varphi_M(\bar{u}_T \circ \mu^s)(A_i) = \varphi_M(\bar{v} \circ \mu^s)(A_j). \tag{4.18}$$

Now, consider $\bar{A} := \mathcal{I} \setminus \bigcup_{i \in N} A_i$. By **D3** and (4.2), $\hat{\mu}^s\left(\bar{A}\right) = \mathbf{0}_N \in \mathbb{R}^N$. Hence by (4.6), (4.7), and (4.1),

$$\varphi_M(\bar{u}_T \circ \mu^s)\left(\bar{A}\right) = 0. \tag{4.19}$$

By (3.3), **D3**, and **D4**, the efficency of the Mertens value entails

$$\bar{u}_T \circ \mu^s (\mathcal{I}) = \min_T (s) .$$

Hence, (4.14), (4.18), and (4.19) entail

$$\varphi_M (\bar{u}_T \circ \mu^s) (A_i) = \begin{cases} 0, & i \in N \backslash \operatorname{argmin}_T (s), \\ \dfrac{s_i}{|\operatorname{argmin}_T (s)|}, & i \in \operatorname{argmin}_T (s), \end{cases} \qquad i \in N. \quad (4.20)$$

Further, (4.9) gives

$$\varphi_M (\bar{v} \circ \mu^s) (A_i) = s_i \cdot \sum_{T \subseteq N : i \in \operatorname{argmin}_T (s)} \frac{\lambda_T (v)}{|\operatorname{argmin}_T (s)|}. \quad (4.21)$$

For $i \in N$, $s \in \mathbb{R}_+^N (\rho)$, $\rho \in R(N)$, by (4.5) and (4.9), (4.21)

$$\varphi_M (\bar{v} \circ \mu^s) (A_i) = \sum_{T \subseteq K_i(\rho), i \in T} \min_T (s) \cdot \lambda_T (v) = s_i \cdot \sum_{T \subseteq K_i(\rho), i \in T} \lambda_T (v) = s_i \cdot MC_i^v (\rho) . \quad (4.22)$$

The importance of the afore mentioned formula lies in the fact that all what matters in the following are the payoffs for

$$s \in \mathbb{R}_+^N (R) := \bigcup_{\rho \in R(N)} \mathbb{R}_+^N (\rho) .$$

For the sake of completeness, we povide the a formula for general $s \in \mathbb{R}_+^N$. We have

$$\varphi_M (\bar{v} \circ \mu^s) (A_i) = \frac{1}{|R(N,s)|} \sum_{\rho \in R(N,s)} s_i \cdot MC_i^v (\rho) , \qquad i \in N, \ s \in \mathbb{R}_+^N, \quad (4.23)$$

where

$$R(N,s) = \{\rho \in R(N) \,|\, \forall i,j \in N : \rho(i) > \rho(j) \iff s_i < s_j\} , \qquad s \in \mathbb{R}_+^N. \quad (4.24)$$

First note that (4.21) is special case of (4.23) because $R(N,s) = \rho$ for $s \in \mathbb{R}_+^N (\rho)$. Since φ_M is linear in v, it suffices to show (4.23) for u_T, $T \subseteq N$, $N \neq \emptyset$. For $\rho \in R(N, u_T)$, the marginal contributions $MC_i^{u_T} (\rho)$ are either 0 or 1. By (4.24), the marginal contribution $MC_i^{u_T} (\rho)$ is 1 iff i is the last player from $\operatorname{argmin}_T (s)$.

Since by (4.24) the probability being the last one from $\operatorname{argmin}_T(s)$ in ρ is the same for all $i \in \operatorname{argmin}_T(s)$, we have

$$\frac{1}{|R(N,s)|} \sum_{\rho \in R(N,s)} s_i \cdot MC_i^v(\rho) = \frac{s_i}{|\operatorname{argmin}_T(s)|} = \varphi_M(\bar{u}_T \circ \mu^s)(A_i),$$

which proves the claim.

5. Replicator dynamics derived from TU games

5.1. Basic idea

The basic idea of this section is to interpret the payoffs

$$f_i^v(s) := \varphi_M(\bar{v} \circ \mu^s)(A_i) = s_i \cdot \sum_{T \subseteq N : i \in \operatorname{argmin}_T(s)} \frac{\lambda_T(v)}{|\operatorname{argmin}_T(s)|}, \tag{5.1}$$

$i \in N,\ v \in \mathbb{V}(N),\ s \in \mathbb{R}_+^N$ as fitness of the populations of the players' agents which determine the velocities of their growth in time. Let

$$f^v : \mathbb{R}_+^N \to \mathbb{R}^N, \qquad s \mapsto (f_i^v(s))_{i \in N}, \qquad s \in \mathbb{R}_+^N.$$

For some fixed $v \in \mathbb{V}(N)$, we obtain the following replicator dynamics for the absolut population sizes given by the ordinary differential equation

$$\dot{s} = \frac{ds}{dt} = f^v(s), \qquad s \in \mathbb{R}_+^N. \tag{5.2}$$

For population shares, we obtain

$$\dot{x} = \frac{dx}{dt} = \varphi^v(x), \qquad x \in \Delta_+^N, \tag{5.3}$$

where $\varphi^v : \Delta_+^N \to \mathbb{R}^N$, $x \mapsto (\varphi_i^v(x))_{i \in N}$ is given by

$$\varphi_i^v(x) := f_i^v(x) - x_i \sum_{j \in N} f_j^v(x), \qquad i \in N,\ x \in \Delta_+^N. \tag{5.4}$$

One easily checks that (5.3) is a replicator dynamic.

5.2. Solutions to the differential equations

By (5.1) and (5.4), the right-hand sides of the ODE (5.2) and (5.3) are discountinuous in general. Hence, the standard results on ODE (see e.g. Weibull 1995, Section 6) do not apply. Instead, we employ the Filippov (1988) approach later on (see Section 5.3)

While the ODE (5.2) and (5.3) are discountinuous in general, it is immediate, again from (5.1) and (5.4), that they are Lipschitz continuous on certain subdomains of $\mathbb{R}_{+}^{N}$ or Δ_{+}^{N}, respectively, on $\mathbb{R}_{+}^{N}(\rho)$ and

$$\Delta_{+}^{N}(\rho) := \Delta_{+}^{N} \cap \mathbb{R}^{N}(\rho), \qquad \rho \in R(N).$$

Hence, Weibull (1995, Theorem 6.1) guarantees the existence of unique (local) solutions Weibull (1995, Definition 6.1) of the ODE (5.2) and (5.3) on

$$\mathbb{R}_{++}^{N}(\rho) := \mathbb{R}_{++}^{N} \cap \mathbb{R}^{N}(\rho) \qquad \text{and} \qquad \Delta_{++}^{N}(\rho) := \Delta_{++}^{N} \cap \mathbb{R}^{N}(\rho)$$

through any point $s^0 \in \mathbb{R}_{++}^{N}(\rho)$ and $x^0 \in \Delta_{++}^{N}(\rho)$, respectively. These solutions are either global or reach the boundary of the domain in finite time.

In particular, the right-hand side of the ODE (5.2) is not only linear, but has a rather simple structure. By (4.22), it is just the inner product of the population size vector with the vector of marginal contributions. One easily checks that the unique local solution of the ODE (5.2) on $\mathbb{R}_{++}^{N}(\rho)$ through $s^0 \in \mathbb{R}_{++}^{N}(\rho)$ at $t=0$ is given by

$$s_i\left(t, s^0\right) = s_i^0 \cdot e^{t \cdot MC_i^v(\rho)}, \qquad i \in N,\ t \in T, \tag{5.5}$$

where $T \subseteq \mathbb{R}$ is an open interval containing 0. Hence, the unique local solution of the ODE (5.3) on $\Delta_{++}^{N}(\rho)$ through $x^0 \in \Delta_{++}^{N}(\rho)$ at $t=0$ is given by

$$x_i\left(t, x^0\right) = \frac{x_i^0 \cdot e^{t \cdot MC_i^v(\rho)}}{\sum_{j \in N} x_j^0 \cdot e^{t \cdot MC_j^v(\rho)}}, \qquad i \in N,\ t \in T. \tag{5.6}$$

Using (5.5 or (5.6), one can dertermine which domain of (potential) dicontinuity is reached when one starts at $s^0 \in \mathbb{R}_{++}^{N}(\rho)$ or $x^0 \in \Delta_{++}^{N}(\rho)$. An ordered partition $(\mathcal{P}, \pi)$ on N is a partition $\mathcal{P} \in \mathbb{P}(N)$ together with a bijection $\pi : \mathcal{P} \to \{1, \ldots, |\mathcal{P}|\}$. Let $\mathbb{P}^{o}(N)$ denote the set of all ordered partitions of N.

For $(\mathcal{P},\pi)\in\mathbb{P}^{o}(N)$, set

$$\begin{aligned}
\mathbb{R}^{N}(\mathcal{P},\pi) &:= \left\{x\in\mathbb{R}^{N}|\forall i,j\in N: x_i\le x_j \iff \pi(\mathcal{P}(i))\ge\pi(\mathcal{P}(j))\right\},\\
\mathbb{R}^{N}_{+}(\mathcal{P},\pi) &:= \mathbb{R}^{N}(\mathcal{P},\pi)\cap\mathbb{R}^{N}_{+},\\
\mathbb{R}^{N}_{++}(\mathcal{P},\pi) &: = \mathbb{R}^{N}(\mathcal{P},\pi)\cap\mathbb{R}^{N}_{++},\\
\Delta^{N}_{+}(\mathcal{P},\pi) &= \mathbb{R}^{N}(\mathcal{P},\pi)\cap\Delta^{N}_{+},\\
\Delta^{N}_{++}(\mathcal{P},\pi) &= \mathbb{R}^{N}(\mathcal{P},\pi)\cap\Delta^{N}_{++}.
\end{aligned}$$

For $i,j\in N$, $i\neq j$, $s^0\in\mathbb{R}^N_{++}$, $v\in\mathbb{V}(N)$, and $\rho\in R(N)$, set

$$t\left(i,j,v,s^0,\rho\right)=\begin{cases}+\infty, & \begin{array}{l}MC_i^v(\rho)=MC_j^v(\rho)\\ \vee\left[\left(MC_i^v(\rho)-MC_j^v(\rho)\right)\left(s_i^0-s_j^0\right)>0\right],\end{array}\\ \dfrac{-\left(\ln s_i^0-\ln s_j^0\right)}{MC_i^v(\rho)-MC_j^v(\rho)}, & \left(MC_i^v(\rho)-MC_j^v(\rho)\right)\left(s_i^0-s_j^0\right)<0.\end{cases}$$

Further, for $s^0\in\mathbb{R}^N_{++}$, $v\in\mathbb{V}(N)$, and $\rho\in R(N)$, let

$$t^*\left(v,s^0,\rho\right)=\min_{i,j\in N:i\neq j}t\left(i,j,v,s^0,\rho\right).$$

By (5.5 or (5.6), $t^*(v,s^0,\rho)$ is the time when the forward solution reaches the boundary of $\mathbb{R}^N_{++}(\rho)$ or $\Delta^N_{++}(\rho)$, respectively. Note that if $t^*(v,s^0,\rho)=+\infty$, then the trajectory stays in $\Delta^N_{++}(\rho)$ for $t\ge 0$.

For $s^0\in\mathbb{R}^N_{++}$, $v\in\mathbb{V}(N)$, and $\rho\in R(N)$, set $\mathcal{P}(v,s^0,\rho)\in\mathbb{P}(N)$:

$$\mathcal{P}\left(v,s^0,\rho\right)(i)=\{i\}\cup\left\{j\in N\setminus\{i\}\,|t\left(i,j,v,s^0,\rho\right)=t^*\left(v,s^0,\rho\right)\neq+\infty\right\},\qquad i\in N,$$

and let $\pi(v,s^0,\rho)$ be the bijection $\mathcal{P}(v,s^0,\rho)\to\{1,\ldots,|\mathcal{P}(v,s^0,\rho)|\}$ given by

$$\begin{aligned}
\pi\left(v,s^0,\rho\right)\left(\mathcal{P}\left(v,s^0,\rho\right)(i)\right) &\ge \pi\left(v,s^0,\rho\right)\left(\mathcal{P}\left(v,s^0,\rho\right)(j)\right)\\
&\iff s_i\left(t^*\left(v,s^0,\rho\right),s^0\right)\le s_j\left(t^*\left(v,s^0,\rho\right),s^0\right),
\end{aligned}$$

$i,j\in N$.

$$\begin{aligned}
s_i^0\cdot e^{t\cdot MC_i^v(\rho)} &= s_j^0\cdot e^{t\cdot MC_j^v(\rho)}\\
\frac{s_i^0}{s_j^0}\cdot e^{t\cdot\left(MC_i^v(\rho)-MC_j^v(\rho)\right)} &= 1\\
\left(\ln s_i^0-\ln s_j^0\right)+t\cdot\left(MC_i^v(\rho)-MC_j^v(\rho)\right) &= 0\\
t &= \frac{-\left(\ln s_i^0-\ln s_j^0\right)}{MC_i^v(\rho)-MC_j^v(\rho)}=
\end{aligned}$$

5.3. Solutions to differential equations with discontinuous right-hand sides: General approach

Filippov (1988, Paragraph 4) considers the following setup.[3]

1. Let $G \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$.
2. Let $f : G \to \mathbb{R}^n$ be piecewise continuous, i.e., G consist of
    1. a finite number $\ell \in \mathbb{N}$ of domains $G_i \subseteq G$, $i = 1, \ldots, \ell$,
    2. and a set $M \subseteq G$ of measure zero, which consists of the boundary points of the G_i, $i = 1, \ldots, \ell$, and
3. f is continuous on the inner of any $G_i \subseteq G$, $i = 1, \ldots, \ell$,
4. for $i = 1, \ldots, \ell$, f on G_i can be extended continuously to the boundary of G_i, i.e., for any sequence $(x_k)_{k\in\mathbb{N}}$ in G_i such that $x_k \to x \in M$, the sequence $(f(x_k))_{k\in\mathbb{N}}$ coverges to some $x \in \mathbb{R}^n$.
5. M consists of a finite number of hyperplanes.

From the point-valued vector field $f : G \to \mathbb{R}^n$, one obtains the set-valued vector field $F : G \rightrightarrows \mathbb{R}^n$,

$$F(x) = \begin{cases} \{f(x)\}, & x \in G \backslash M, \\ \operatorname{con}\{\lim_{x_k \in G_i : x_k \to x} f(x_k) \,|\, i = 1, \ldots, \ell,\ x \in \partial G_i\} & x \in M, \end{cases}$$

where $\operatorname{con}(A)$ and $\partial(A)$ denote the convex hull and the boundary of $A \subseteq \mathbb{R}^n$ respectively.

A solution to the ODE $\dot{x} = f(x)$ is a solution of the differential inclusion $\dot{x} \in F(x)$, i.e., an absolutely continuous function $x : I \to \mathbb{R}^n$ on some interval $I \subseteq \mathbb{R}$ containing 0 for which $\dot{x}(t) \in F(x)$ almost everywhere on I.

Filippov (1988, pp. 81) presents the following results on the solutions to ODE as above:

A. Through any point of the interior of G there passes a solution.

B. Each solution lying within a given closed bounded domain is continued on both sides to reach the boundary of the domain.

[3] Actually, we restrict attention to autonomous ODE with discontinuous right-hand sides. Further, we focus on hypersurfaces that are hyperplanes.

5.4. Solutions to differential equations with discontinuous right-hand sides: Population dynamic

Fortunately, the ODE (5.2) and (5.3) fit the setup outlined in Section 5.3. The domains of continuity are $\left(\mathbb{R}_{+}^{N}(\rho)\right)_{\rho\in R(N)}$ and $\left(\Delta_{+}^{N}(\rho)\right)_{\rho\in R(N)}$, respectively; the domains of (potential) discontinuity are

$$\left(\mathbb{R}_{+}^{N}(\mathcal{P},\pi)\right)_{(\mathcal{P},\pi)\in\mathbb{P}^{o}(N)} \qquad \text{and} \qquad \left(\Delta_{+}^{N}(\mathcal{P},\pi)\right)_{(\mathcal{P},\pi)\in\mathbb{P}^{o}(N)},$$

respectively. By (4.22), (5.1), and (5.4), the functions f^{v} and φ^{v} are continous up to the boundary on any of their domains of continuity. Hence by Filippov (1988, pp. 81), we know that through any $s^{0}\in\mathbb{R}_{++}^{N}$ and $x^{0}\in\Delta_{++}^{N}$ runs a solution of the respective ODE (A). Since our ODE are autonomous, any such solution is global (B).

Conjecture 5.1. *Whenever a trajectory leaves $\mathbb{R}_{++}^{N}(\rho)$, it never returns to $\mathbb{R}_{++}^{N}(\rho)$.*

6. Stability of population shares

6.1. General stuff

- We are interested in (Lyapunov, asymptotically) stable population shares vectors under the replicator dynamic (5.3).
- To deal with the possible non-uniqueness of solutions to (5.3), we suggest the following modification of stability.
- $x^{*}\in\Delta^{N}$ is Lyapunov stable in the ODE (5.3) if every neighborhood U of x^{*} there is a neighborhood U^{0} of x^{*}, $U^{0}\subseteq U$ such that for any solution ξ to (5.3), $\xi(x^{0},t)\in U$ for all $x^{0}\in U^{0}\cap\Delta^{N}$ and $t\geq 0$.
- $x^{*}\in\Delta^{N}$ is asymptotically stable in the ODE (5.3) if x^{*} is Lyapunov stable in the ODE (5.3) and there is some neighborhood U of x such that for all $x\in U\cap\Delta^{N}$ and all solutions ξ of (5.3), we have

$$\lim_{t\to\infty}\xi(x,t)=x^{*}.$$

6.2. Conjectures

Conjecture 6.1. *If* (N, v) *is strictly convex then* $\left(\frac{1}{|N|}, \ldots, \frac{1}{|N|}\right)$ *is the unique asymptocically stable population share vector.*

Conjecture 6.2. *If* (N, v) *is strictly concave then any asymptocically stable population share vector lies in some domain of continuity.*

Conjecture 6.3. *If* (N, v) *is a simple game with the set of minimal winning coalitions* $\mathbb{M}$*, the asymptotically stable states* $\hat{x} = (\hat{x}_1, ..., \hat{x}_n)$ *are characterized by minimal winning coalitions* $W \in \mathbb{M}$ *and*

$$\hat{x}_i = \begin{cases} \frac{1}{|W|}, & i \in W \\ 0, & \text{otherwise} \end{cases}$$

The apex game discussed in the introduction corroborates this conjecture.

In ENGT, strictly dominated strategies are wielded out. We present a dominance definition and show that we do not have a similar result in ECGT.

Definition 6.4. *Let* $v \in \mathbb{V}(N)$*. Player* $i \in N$ *strictly dominates player* $j \in N$ *if* $v(K \cup \{i\}) > v(K \cup \{j\})$ *holds for all* $K \subseteq N \backslash \{i, j\}$*. Player* $i \in N$ *weakly dominates player* $j \in N$ *if* $v(K \cup \{i\}) \geq v(K \cup \{j\})$ *holds for all* $K \subseteq N \backslash \{i, j\}$ *and if there is a coalition* $\hat{K} \in N \backslash \{i, j\}$ *such that* $v\left(\hat{K} \cup \{i\}\right) > v\left(\hat{K} \cup \{j\}\right)$ *is true. In that case, we also say that* i *weakly dominates* j *with strong* $\hat{K}$*-dominance.*

Note that strict and weak dominance are equivalent for $n = 2$.

Conjecture 6.5. *A strictly dominated player does not need to vanish. A null player does not need to vanish.*

We conjecture that a strictly dominated null player vanishes. However, neither one of these two conditions are sufficient for driving out a player. The first assertion follows from the game given by $N = \{1, 2\}$, $v(1) = 1, v(2) = 0$ and $v(1, 2) = 3$. Assume the initial population share vector $x(0) = \left(\frac{4}{5}, \frac{1}{5}\right)$. Player 2 is strictly dominated but holds his ground as can be seen in fig. 6.1. Also, a weakly dominating player (as the apex player) can vanish while the player dominated by him does not, as we have seen in the introduction (see fig. 1.2).

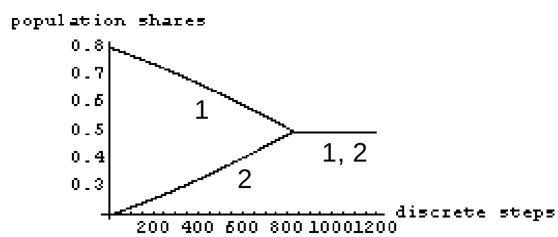

Figure 6.1: Player 2 is dominated but does not vanish.

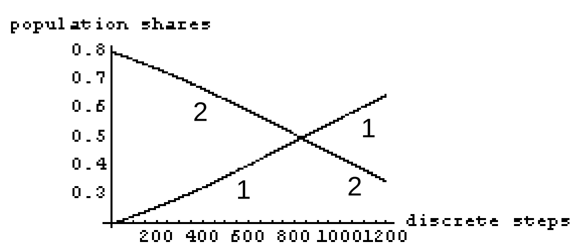

Figure 6.2: Player 1 is a dominating null player.

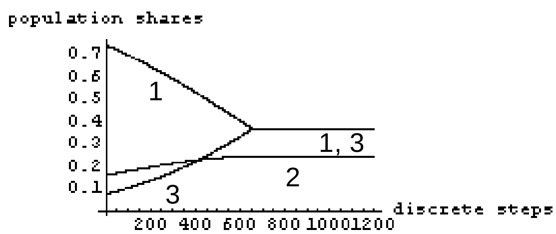

Figure 6.3: Different non-zero shares in the long run

The second asssertion can be seen from the game for two players given by $v(1) = 0$, $v(2) = -1 = v(1,2)$. Fig. 6.2 (with $x(0) = \left(\frac{1}{5}, \frac{4}{5}\right)$) demonstrates that a null player (player 1 in our case) does not need to vanish.

The examples presented so far may give the impression that we cannot have different non-zero shares in the long run. However, this is not true. Consider $N = \{1,2,3\}$, $v \in V(N)$ given by $v(1) = v(2) = v(3) = 0$, $v(1,3) = 2$, $v(1,2) = v(2,3) = 1$ and $v(1,2,3) = 3$. The initial population share vector $x(0) = \left(\frac{3}{4}, \frac{1}{6}, \frac{1}{12}\right)$ yields fig. 6.3.

7. Conclusion

Our paper presents work in progress. We still need to work on the solutions to our differential equations.

With respect to future research, note that the replicator dynamics are concerned with selection. Of course, mutation is the other evolutionary force to be reckoned with. It is concerned with the change of parameters rather than the selection pressures for a given set of parameters. Within our framework, mutation can take different forms:

1. We may consider small changes of the coalition function v.

2. Other players could be added with very small sizes such that the worths for the other players stays the same for a zero size of the new arrival.

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