A note on the Chun characterization of the Shapley value

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Abstract

In this note, we revisit the Chun (1989, Games Econ Behav 1: 119–130) characterization of the Shapley value via efficiency, the Null game property, coalitional strategic equivalence, and fair ranking. In particular, we show that coalitional strategic equivalence and marginality (Young, 1985, Int J Game Theory 14: 65–72) are equivalent. Using this fact and removing an inconsistency of fair ranking, one obtains a new characterization of the Shapley value.

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1. Introduction

The most influential one-point solution concept for TU games certainly is the value introduced and characterized by Loyd S. Shapley (1953). Later, a standard axiomatization via efficiency, additivity, symmetry, the Null player property was derived from Shapley's original characterization (Aumann, 1989). Besides this standard axiomatization, numerous alternative characterizations have been suggested, in particular, characterizations that do without the additivity axiom, e.g. those of Myerson (1980), Young (1985), Chun (1989), and van den Brink (2001). Though additivity is a nice mathematical property and though it may be justified by the Roth (1977) arguments, it seems to be desirable to avoid additivity, because it does not refer to fairness considerations.

Besides efficiency and symmetry, Young (1985) employs the very elegant *marginality* axiom, which he calls *independence.* Yet, as Chun (1989), we feel that *marginality* is much more appropriate. Chun (1989) characterizes the Shapley value by efficiency, the Null game axiom, *coalitional strategic equivalence*, and *fair ranking.*

Marginality and coalitional strategic equivalence seem to be close relatives. While Chun (1989, Lemma 2) shows that marginality entails coalitional strategic equivalence, his discussion in Section 4 suggests that the inverse implication does not hold in general. In this note, however, we find the latter not to be true. Indeed, it is quite easy to see that both properties are equivalent. Hence, some of the assertions on the examples in Chun's Section 4 must be wrong.

Further, we feel that fair ranking is somewhat inconsistent. Fair ranking requires that the ranking of the payoffs of two players is not affected by any changes of the coalition function for a coalition containing both of them. But then, the following question comes to mind: Why shouldn't this equally hold if the change concerns a coalition which does *not* contain both players? Strengthening fair ranking in this respect and then relaxing it to hold only for equal payoffs gives a weak version of coalitional independence (Hernández Lamoneda, Juárez Garcìa and Sánchez Sánchez, 2005). Using this axiom within the Chun characterization instead of fair ranking, symmetry can easily be inferred. Together with the fact that marginality and coalitional strategic equivalence are equivalent, one obtains an alternative characterization of the Shapley value.

The plan of this note is as follows: Basic definitions and notation are given in second section. In the third section, we reconsider coalitional strategic equivalence. The fourth section deals with the fair ranking axiom.

2. Basic definitions and notation

A TU game is a pair (N, v) consisting of a non-empty and finite set of players N and the coalition function $v \in V(N) := \{f : 2^N \to \mathbb{R}, v(\emptyset) = 0\}$. Subsets of N are called coalitions, and $v(K)$ is called the worth of coalition K. For $T \in 2^N \backslash \{\emptyset\}$, the game (N, u_T), $u_T(K) = 1$ if $T \subseteq K$ and $u_T(K) = 0$ otherwise, is called a unanimity game. The sum $v + w$ and product $\lambda \cdot v$ with a scalar $\lambda \in \mathbb{R}$, $v, w \in V(N)$ are given by $(v + w)(K) = v(K) + w(K)$ and $(\lambda \cdot v)(K) = \lambda \cdot v(K)$ for all $K \subseteq N$. It is well known that any $v \in V(N)$ can be uniquely represented by unanimity games,

$$v = \sum_{T \in 2^N \backslash \{\emptyset\}} \lambda_T(v) \cdot u_T, \qquad \lambda_T(v) \in \mathbb{R}, \tag{1}$$

where the Harsanyi (1959) dividends $\lambda_T(v)$ are given inductively by

$$\lambda_{\{i\}}(v) = v(\{i\}) \quad \text{and} \quad \lambda_K(v) = v(K) - \sum_{\emptyset \neq K \subsetneq T} \lambda_K(v), \quad \emptyset \neq K \subseteq N. \tag{2}$$

The Null game on N is denoted $(N, \mathbf{0})$ where $\mathbf{0}(K) = 0$ for all $K \subseteq N$. The marginal contribution of $i \in N$ to $K \subseteq N \backslash \{i\}$ is given by $MC_i^v(K) := v(K \cup \{i\}) - v(K)$. A player i is called a Dummy player (Null player) iff $MC_i^v(K) = v(\{i\})$ $(= 0)$ for all $K \subseteq N \backslash \{i\}$; Players $i, j \in N$ are called symmetric if $MC_i^v(K) = MC_j^v(K)$ for all $K \subseteq N \backslash \{i, j\}$. A value is an operator φ that assigns payoff vectors to all games, $\varphi(N, v) \in \mathbb{R}^N$.

An order of a set N is a bijection $\sigma : N \to \{1, \ldots, |N|\}$ with the interpretation that i is the $\sigma(i)$th player in σ. The set of these orders is denoted by $\Sigma(N)$. The set of players weakly preceding i in σ is denoted by $K_i(\sigma) = \{j : \sigma(j) \leq \sigma(i)\}$. The marginal contribution of i under σ is denoted $MC_i^v(\sigma) := MC_i^v(K_i(\sigma) \backslash \{i\})$. The Shapley (1953) value Sh,

$$\text{Sh}_i(N, v) := \frac{1}{|\Sigma(N)|} \sum_{\sigma \in \Sigma(N)} MC_i^v(\sigma), \qquad i \in N, \tag{3}$$

is characterized by the well-known axioms **E**, **S**, **N**, and **A** below.

Efficiency, E. $\sum_{i \in N} \varphi_i(N, v) = v(N)$.

Symmetry, S. If $i, j \in N$ are symmetric then $\varphi_i(N, v) = \varphi_j(N, v)$.

Null player, N. If $i \in N$ is a Null player then $\varphi_i(N,v) = 0$.

Additivity, A. $\varphi(N, v + v') = \varphi(N, v) + \varphi(N, v')$.

The Young (1985, Theorem 2) characterization involves **E**, **S**, and **M**.

Marginality, M. If $MC_i^v(K) = MC_i^w(K)$ for all $K \subseteq N \setminus \{i\}$ then $\varphi_i(N,v) = \varphi_i(N,w)$.

Chun (1989, Theorem 3) provides a characterization by **E**, **NG**, **CSE**, and **FR**.

Null game, NG. For all $i \in N$, $\varphi_i(N, \mathbf{0}) = 0$.

Coalitional strategic equivalence, CSE. If $i \in N$, $\lambda \in \mathbb{R}$, and $i \notin T \in 2^N \setminus \{\emptyset\}$ then $\varphi_i(N,v) = \varphi_i(N, v + \lambda \cdot u_T)$.

Fair ranking, FR. If $i, j \in T \subseteq N$ and $v(S) = w(S)$ for all $T \neq S \subseteq N$, then $\varphi_i(N,v) > \varphi_j(N,v)$ iff $\varphi_i(N,w) > \varphi_j(N,w)$.

3. Coalitional strategic equivalence versus marginality

While Chun (1989, Lemma 2) correctly observes that **M** implies **CSE**, he is wrong concerning the converse implication, which is implicit in the discussion of the values ϕ^2 and ϕ^4 in his Section 4. In order to prepare the proof of the equivalence of **M** and **CSE**, we first provide a simple lemma which establishes the relation between the hypothesis of **M** and the Harsanyi dividends.

Lemma 1. *$MC_i^v(K) = MC_i^w(K)$ for all $K \subseteq N \setminus \{i\}$ iff $\lambda_{K \cup \{i\}}(v) = \lambda_{K \cup \{i\}}(w)$ for all $K \subseteq N \setminus \{i\}$.*

Proof. From (1), it is clear that $MC_i^v(K) = \sum_{T \subseteq K} \lambda_{T \cup \{i\}}(v)$ for all $i \notin K \subseteq N$. By induction on $|K|$, the claim is immediate. ∎

Proposition 2. ***CSE** and **M** are equivalent.*

Proof. By Chun (1989, Lemma 2), **M** implies **CSE**. The other way round, let $i \in N$, v, and w satisfy the hypothesis of **M**, and let φ obey **CSE**. By Lemma 1, we have

$$v - \sum_{\emptyset \neq T \subseteq N \setminus \{i\}} \lambda_T(v) \cdot u_T =: q := w - \sum_{\emptyset \neq T \subseteq N \setminus \{i\}} \lambda_T(w) \cdot u_T.$$

Successive application of **CSE** on both sides of this equation yields $\varphi_i(N,v) = \varphi_i(N,q) = \varphi_i(N,w)$, i.e., **CSE** implies **M**. ∎

Proposition 2 then entails that the claim that the value ϕ^2 given by

$$\phi_i^2(N,v) = \sum_{i\in S\subseteq N:|S|=2} v(S) - \sum_{j\in N} v(\{j\}), \qquad i \in N$$

satisfies **CSE** but not **M** (Chun, 1989, p. 128) cannot be true. Indeed, ϕ^2 meets both properties, which can be seen from the following formulations of ϕ^2,

$$\begin{aligned}\phi_i^2(N,v) &= \sum_{j\in N\setminus\{i\}} MC_i^v(\{j\}) - MC_i^v(\emptyset) \\ &= (|N|-2)\cdot\lambda_{\{i\}}(v) + \sum_{j\in N\setminus\{i\}} \lambda_{\{i,j\}}(v), \qquad i \in N.\end{aligned}$$

Also, the same assertion for the values ϕ^4 (Chun, 1989, p. 129) is wrong: Consider $N = \{1,2\}$, $v = u_N + u_{\{1\}} - u_{\{2\}}$, $w = u_N + u_{\{1\}}$. Let some of the values ϕ^4 be given by

$$\phi_i^4(N,q) = \begin{cases} \alpha_i\cdot\lambda_N(q) + \lambda_{\{i\}}(q), & \lambda_{\{1\}}(q) + \lambda_{\{2\}}(q) \neq 0, \\ \frac{1}{2}\cdot\lambda_N(q) + \lambda_{\{i\}}(q), & \lambda_{\{1\}}(q) + \lambda_{\{2\}}(q) = 0, \end{cases}$$

for $i \in N$, where $\alpha_1 = \frac{1}{3}$ and $\alpha_2 = \frac{2}{3}$. Then, i, v, and w satisfy the hypotheses of **CSE** or **M**, but we have

$$\phi_1^4(N,v) = \frac{1}{2}\cdot 1 + 1 \neq \frac{1}{3}\cdot 1 + 1 = \phi_1^4(N,v).$$

4. Fair ranking versus balanced fair ranking

In a sense, the fair ranking axiom **FR** is unbalanced. While it is intuitive to require that point-changes of the coalition function at some coalition T containing both i and j should not affect these players' ranking of payoffs, it is not too intuitive not also to require this property in the case that both i and j are not members of T, i.e., not to require the following axiom to hold.

Lower fair ranking, LFR. If $i,j \notin T$ and $v(S) = w(S)$ for all $T \neq S \subseteq N$, then $\varphi_i(N,v) > \varphi_j(N,v)$ iff $\varphi_i(N,w) > \varphi_j(N,w)$.

Even stronger, one could argue that **LFR** is more innocuous than **FR**. In view of Proposition 2, the axiom **FR** in the Chun characterization takes the place of the axiom **S** in the Young characterization. Because neither **LFR** nor **FR** trigger changes that affect the hypothesis of **S** and because both are substitutes for **S**, it is more intuitive to combine them. Therefore, we strengthen **FR** by **LFR**, which gives a balanced fair ranking axiom, **BFR**.

Balanced fair ranking, BFR. If $i,j \in T \subseteq N$ or $i,j \notin T$, and $v(S) = w(S)$ for all $T \neq S \subseteq N$, then $\varphi_i(N,v) > \varphi_j(N,v)$ iff $\varphi_i(N,w) > \varphi_j(N,w)$.

In the following, we employ some relaxation of **BFR**. Instead of the preservation of the ranking of the players' payoffs, only the equality of payoffs is required to be unaffected by alterations of the coalition function as in **BFR**. The resulting axiom is called weak coalitional independence, **WCI**. This name is justified by the structural similarity to the coalitional independence axiom (Hernández Lamoneda et al., 2005), **CI**, and observation that **FR**, **LFR**, **BFR**, and **WCI** are implied by **CI**, which is easy to check. Note that **FR** neither implies nor is implied by **WCI**.

Weak coalitional independence, WCI. If $v(S) = w(S)$ for all $T \neq S \subseteq N$ and $i,j \in T$ or $i,j \notin T$, then $\varphi_i(N,v) = \varphi_j(N,v)$ iff $\varphi_i(N,w) = \varphi_j(N,w)$.

Coalitional independence, CI. If $v(S) = w(S)$ for all $T \neq S \subseteq N$ and $i,j \in T$ or $i,j \notin T$, then $\varphi_i(N,v) - \varphi_j(N,v) = \varphi_i(N,w) - \varphi_j(N,w)$.

We now show that one can replace **FR** by **WCI** in the Chun characterization of the Shapley value. The main work is done by the following proposition, which directly infers **S** from the altered set of axioms.

Proposition 3. ***E**, **NG**, **CSE**, and **WCI** imply **S**.*

Proof. Let φ satisfy **E**, **NG**, **CSE** and **WCI**, and let i and j be symmetric in (N,v). We proceed in a number of steps.

Step 1. Obviously, **E**, **NG**, and **CSE** imply $\varphi_i\left(N, \lambda \cdot u_{\{i\}}\right) = \lambda$.

Step 2. If i is a Dummy player in (N,v), then, by (1), $\lambda_T(v) = 0$ if $i \in T$ and $|T| > 1$, $T \subseteq N$. Hence, i, v, and $\lambda_{\{i\}}(v) \cdot u_{\{i\}}$ satisfy the hypothesis of **M**, which by Proposition 2 is equivalent to **CSE**. Together with *Step 1* and (2), this implies $\varphi_i(N,v) = \lambda_{\{i\}}(v) = v(\{i\})$.

Step 3. Define $w \in V(N)$ as follows:

$$w(K) = \begin{cases} v(K \cup \{i\}) - v(\{i\}), & \emptyset \neq K \subseteq N \setminus \{i,j\}, \\ v(K \setminus \{i\}) + v(\{i\}), & i,j \in K \subseteq N, \\ v(K), & \text{else.} \end{cases}$$

Since i and j are symmetric, this implies $MC_i^w(K) = v(\{i\}) = v(\{j\}) = MC_j^w(K')$ for all $K \subseteq N \setminus \{i\}$ and $K' \subseteq N \setminus \{j\}$, i.e., i and j are Dummy players in (N,w). Together with *Step 2*, this implies $\varphi_i(N,w) = v(\{i\}) = v(\{j\}) = \varphi_j(N,w)$.

Step 4. Further, it is clear that one obtains w from v by successive transformations which satisfy the hypothesis of **WCI**. Hence, $\varphi_i(N, v) = \varphi_j(N, v)$ iff $\varphi_i(N, w) = \varphi_j(N, w)$. Using *Step 3* one derives $\varphi_i(N, v) = \varphi_j(N, v)$, which proves the claim. ∎

Together with Proposition 2, one obtains the following corollary to Young (1985, Theorem 2).

Corollary 4. *The Shapley value is characterized by **E**, **NG**, **CSE**, and **WCI**.*

Proof. It is well known that the Shapley value satisfies **E**, **NG**, and **CSE**; **WCI** is immediate from (3). Let φ satisfy **E**, **NG**, **CSE** and **WCI**. By Propositions 2 and 3, φ satisfies **M** and **S**. Young (1985, Theorem 2) implies that φ is the Shapley value. ∎

References

Aumann, R. J. (1989). *Lectures on Game Theory*, Westview Press, Boulder et al.

Chun, Y. (1989). A new axiomatization of the Shapley value, *Games and Economic Behavior* **1**: 119–130.

Harsanyi, J. C. (1959). A bargaining model for cooperative n-person games., *in* A. W. Tucker and R. D. Luce (eds), *Contributions to the Theory of Games IV*, Vol. 2, Princeton University Press, Princeton NJ, pp. 325–355.

Hernández Lamoneda, L., Juárez Garcìa, R. and Sánchez Sánchez, F. (2005). Solutions without dummy axiom for TU cooperative games, *Technical report*. presented at SING1.

Myerson, R. B. (1980). Conference structures and fair allocation rules, *International Journal of Game Theory* **9**: 169–182.

Roth, A. E. (1977). The Shapley value as a von Neumann-Morgenstern utility, *Econometrica* **45**: 657–664.

Shapley, L. S. (1953). A value for n-person games, *in* H. Kuhn and A. Tucker (eds), *Contributions to the Theory of Games*, Vol. II, Princeton University Press, Princeton, pp. 307–317.

van den Brink, R. (2001). An axiomatization of the Shapley value using a fairness property, *International Journal of Game Theory* **30**: 309–319.

Young, H. P. (1985). Monotonic solutions of cooperative games, *International Journal of Game Theory* **14**: 65–72.