A characterization of the Owen value without the additivity axiom

André Casajus

(March 2007, this version: February 14, 2008, 14:45)

Abstract

We provide a characterization of the Owen value for TU games with a coalition structure without the additivity axiom.

Journal of Economic Literature Classification Number: C71.

Key Words: Owen value, additivity, marginality, differential marginality.
1. Introduction

Owen (1977) suggests an extension of the Shapley (1953) value to TU games with a coalition structure (henceforth CS-games and CS-values). His original characterization as well as the characterizations of Hart and Kurz (1983), Winter (1992), Hamiache (2001), and Albizuri (2008) involve the additivity axiom. Although mathematically elegant and in spite of the Roth (1977) motivation, additivity hardly can be viewed as an expression of fairness. While the Calvo, Javier Lasaga and Winter (1996) axiomatization does not employ additivity, it involves CS-games with varying player sets.

Only recently, Khmelnitskaya and Yanovskaya (2007) provide a characterization of the Owen value on a fixed player set that does without additivity. Besides efficiency, symmetry within components, and symmetry between components, they employ the marginality axiom, which was used by Young (1985) in his characterization of the Shapley value. Marginality requires a player’s payoff to depend only on his own marginal contributions.

Instead of marginality, we suggest two differential versions of marginality. Differential marginality within components is concerned with two players who inhabit the same component. This axiom requires that the difference of such players’ payoffs is affected only by the differences of their marginal contributions. Similarly, differential marginality between components deals with the difference of the sum of payoffs of two components. The difference of these sums should depend only on the difference of the marginal contributions of the components in the intermediate game, i.e., in the game between the components.

There are—at least—two reasons to consider differential versions of marginality. The first one are solidarity considerations. For example, it might be desirable that a player without any productive contributions participates in the overall worth created. Marginality, however, postulates that payoffs depend only on one’s own performance. Indeed, together with efficiency and symmetry axioms, marginality entails the Null player axiom (see e.g. Young (1985), Khmelnitskaya and Yanovskaya (2007)), which embodies some no-solidarity property. But even though one may wish to have some extent of solidarity among players, one could like to have that performance pays in the following sense. Larger differences of performance should

\footnote{Originally, Young (1985, p. 71) refers to this axiom as an independence condition; following Chun (1989), we call it marginality.}
result in larger differences of payoffs. In technical terms, higher differences of marginal contributions should lead to higher differences of payoffs. Note that differential marginality is somewhat weaker than this requirement. For example, the solidarity value (Nowak and Radzik, 1994) satisfies efficiency and symmetry, but neither the Null player axiom nor marginality. Yet, the solidarity value can be characterized by efficiency, differential marginality, and the A-Null player axiom Casajus (2007, Section 5).

A second reason for considering differential marginality is that marginality combined with other desirable axioms may impede a component efficient CS-value to recognize outside options, i.e., to reflect the players’ productive potential outside their own components (see Casajus (2008, Sections 1–2) for a discussion of outside options and their relation to some axioms). For example, component efficiency, symmetry within components, and marginality characterize the AD-value (Aumann and Drèze, 1974). Since the AD-value is the restriction of the Shapley value to components, it cannot account for outside options. Recently, Wiese (2007) and Casajus (2008) introduce the outside-option value and the $\chi$-value, which both recognize outside options. While obeying component efficiency and symmetry within components, both CS-values do not meet marginality. Instead, they satisfy differential marginality within components. Moreover, one characterization of the $\chi$-value involves differential marginality Casajus (2007, Section 5).

In this note, we show that the Owen value can be characterized using differential marginality within and between components. Besides these axioms, our characterization involves efficiency, the Null player axiom, and the Null component axiom. It turns out that this characterization is independent from the Khmelnitskaya and Yanovskaya (2007) axiomatization. Moreover, a similar characterization can be obtained for the generalization of the Owen value suggested by Winter (1989) for games with a level structure.

The plan of this note is as follows: Basic definitions and notation are given in second section. In the third section, we establish and discuss our characterization. A remark on the Winter (1989) value concludes the note.

2. Basic definitions and notation

A TU game is a pair $(N, v)$ consisting of a non-empty and finite set of players $N$ and the coalition function $v \in V(N) := \{ f : 2^N \to \mathbb{R} | v(\emptyset) = 0 \}$. Subsets of $N$ are called coalitions, and $v(K)$ is called the worth of coalition $K$. The Null game
on \( N \) is denoted \((N, 0)\), where 0 \((K) = 0\) for all \( K \subseteq N \). For \( T \in 2^N \setminus \{\emptyset\} \), the game \((N, u_T)\), \( u_T(K) = 1 \) if \( T \subseteq K \) and \( u_T(K) = 0 \) otherwise, is called a **unanimity game**. Any \( v \in V(N) \) can be uniquely represented by unanimity games,

\[
v = \sum_{T \in 2^N \setminus \emptyset} \lambda_T(v) \cdot u_T , \quad \lambda_T(v) \in \mathbb{R} .
\]

For \( v, w \in V(N) \) and \( \alpha \in \mathbb{R} \), \( v + w \in V(N) \) and \( \alpha \cdot v \in V(N) \) are given by \((v + w)(K) = v(K) + w(K)\) and \((\alpha \cdot v)(K) = \alpha \cdot v(K)\) for all \( K \subseteq N \). A player \( i \in N \) is called a **Null player** in \((N, v)\) iff \( v(K \cup \{ i \}) = v(K) \) for all \( K \subseteq N \setminus \{ i \} \); players \( i, j \in N \) are called symmetric in \((N, v)\) if \( v(K \cup \{ i \}) = v(K \cup \{ j \}) \) for all \( K \subseteq N \setminus \{ i, j \} \). The restriction of \( v \) to \( N' \subseteq N \) is denoted \( v|_{N'} \).

A **coalition structure** on \( N \) is a partition \( \mathcal{P} \subseteq 2^N \) of \( N \); a **CS-game** is a game together with a coalition structure, \((N, v, \mathcal{P})\). The elements of \( \mathcal{P} \) are referred to as components; \( \mathcal{P}(i) \) denotes the component containing player \( i \); \( \mathcal{P}|_K \) \(:= \{ P \cap K | P \in \mathcal{P} \} \setminus \{ \emptyset \} \), \( K \subseteq N \), stands for the restriction of \( \mathcal{P} \) to \( K \). For \( K \subseteq N \), let \( \mathcal{P}(K) \) denote the set \( \bigcup_{i \in K} \mathcal{P}(i) \). A component \( P \in \mathcal{P} \) is called a **Null component** in \((N, v, \mathcal{P})\) iff \( v(\mathcal{P}(K) \cup P) = v(\mathcal{P}(K)) \) for all \( K \subseteq N \setminus P \). Components \( P, P' \in \mathcal{P} \) are called symmetric in \((N, v, \mathcal{P})\) iff \( v(\mathcal{P}(K) \cup P) = v(\mathcal{P}(K) \cup P') \) for all \( K \subseteq N \setminus (P \cup P') \). A **CS-value** is an operator \( \varphi \) that assigns payoff vectors to all CS-games, \( \varphi(N, v, \mathcal{P}) \in \mathbb{R}^N \). For \( K \subseteq N \), we denote \( \sum_{i \in K} \varphi_i(N, v, \mathcal{P}) \) by \( \varphi(N, v, \mathcal{P})(K) \).

An **order** of a set \( N \) is a bijection \( \sigma : N \rightarrow \{1, \ldots, |N|\} \) with the interpretation that \( i \) is the \( \sigma(i) \)th player in \( \sigma \). The set of these orders is denoted by \( \Sigma(N) \). The **marginal contribution** of \( i \in N \) in \( \sigma \in \Sigma(N) \) is defined as \( MC_i^\sigma(\sigma) := v(K_i(\sigma)) - v(K_i(\sigma) \setminus \{ i \}) \), where \( K_i(\sigma) := \{ j \in N | \sigma(j) \leq \sigma(i) \} \). For any coalition structure \( \mathcal{P} \) on \( N \),

\[
\Sigma(N, \mathcal{P}) := \{ \sigma \in \Sigma(N) | \forall P \in \mathcal{P} \land \forall i, j \in P : |\sigma(i) - \sigma(j)| < |P| \}
\]

is the set of orders on \( N \) **compatible** with \( \mathcal{P} \). The **Owen (1977) value** is given by

\[
Ow_i(N, v, \mathcal{P}) := |\Sigma(N, \mathcal{P})|^{-1} \sum_{\sigma \in \Sigma(N, \mathcal{P})} MC_i^\sigma(\sigma) , \quad i \in N .
\]

It is characterized by the axioms **E**, **A**, **CS**, **SC**, and **N** below (Winter, 1992). The Khmelevskiy and Yanovskaya (2007) characterization replaces **A** and **N** by the axiom **M** below. Note that both characterizations work for fixed \((N, \mathcal{P})\).

**Efficiency**, \( \Sigma_{i \in N} \varphi_i(N, v, \mathcal{P}) = v(N) \).
Additivity, A. \( \varphi(N, v + w, \mathcal{P}) = \varphi(N, v, \mathcal{P}) + \varphi(N, w, \mathcal{P}) \).

Null player, N. If \( i \in N \) is a Null player in \((N, v)\), then \( \varphi_i(N, v, \mathcal{P}) = 0 \).

Symmetry within components, CS. If \( i, j \in N \) are symmetric in \((N, v)\) and \( j \in \mathcal{P}(i) \), then \( \varphi_i(N, v, \mathcal{P}) = \varphi_j(N, v, \mathcal{P}) \).

Symmetry between components, SC. If \( \mathcal{P}, \mathcal{P}' \in \mathcal{P} \) are symmetric in \((N, v, \mathcal{P})\), then
\[
\sum_{i \in \mathcal{P}} \varphi_i(N, v, \mathcal{P}) = \sum_{i \in \mathcal{P}'} \varphi_i(N, v, \mathcal{P}) .
\]

Marginality, M. If \( v(K \cup \{i\}) - v(K) = w(K \cup \{i\}) - w(K) \) for all \( K \subseteq N \setminus \{i\} \) then 
\[
\varphi_i(N, v, \mathcal{P}) = \varphi_i(N, w, \mathcal{P}) .
\]

3. An axiomatization of the Owen value via differential marginality

In this section, we suggest three axioms, the Null component axiom, differential marginality within components, and differential marginality between components, which together with efficiency and the Null player axiom characterize the Owen value. Moreover, we show that this characterization is non-redundant. Finally, we argue that our axiomatization is independent from the Khmelnitskaya and Yanovskaya (2007) characterization for fixed player sets and coalition structures.

3.1. The axioms. Loosely speaking, our first axiom requires the Null player axiom to hold for the intermediate game \((\mathcal{P}, v^\mathcal{P})\), where \( v^\mathcal{P} \in V(\mathcal{P}) \) and \( v^\mathcal{P}(K) = v(\bigcup_{P \in K} P) \) for all \( K \subseteq \mathcal{P} \). Note that a Null component in \((N, v, \mathcal{P})\) is a Null player in \((\mathcal{P}, v^\mathcal{P})\).

Null component, NC. If \( P \in \mathcal{P} \) is a Null component in \((N, v, \mathcal{P})\), then
\[
\sum_{i \in P} \varphi_i(N, v, \mathcal{P}) = 0 .
\]

The next axiom is a natural modification of marginality in the context of CS-games. In contrast to M, this axiom is concerned with the difference of payoffs of two players, who inhabit the same component. This difference should be influenced only by the differences of their marginal contributions.

Differential marginality within components, CDM. If \( j \in \mathcal{P}(i) \) and
\[
v(K \cup \{i\}) - v(K \cup \{j\}) = w(K \cup \{i\}) - w(K \cup \{j\})
\]
for $i, j \in N$, $v, w \in V(N)$, and for all $K \subseteq N \setminus \{i, j\}$, then

$$\varphi_i(N, v, P) - \varphi_j(N, v, P) = \varphi_i(N, w, P) - \varphi_j(N, w, P).$$

The following axiom, roughly speaking, results from applying CDM to the intermediate game with the trivial coalition structure.

**Differential marginality between components, DMC.** If

$$v(P(K \cup P) - v(P(K) \cup P') = w(P(K) \cup P) - w(P(K) \cup P')$$

for $P, P' \in \mathcal{P}$, $v, w \in V(N)$, and for all $K \subseteq N \setminus (P \cup P')$, then

$$\sum_{i \in P} \varphi_i(N, v, P) - \sum_{i \in P'} \varphi_i(N, v, P) = \sum_{i \in P} \varphi_i(N, w, P) - \sum_{i \in P'} \varphi_i(N, w, P).$$

### 3.2. The characterization.

The following lemma establishes relations between the symmetry axioms CS and SC and the differential marginality axioms CDM and DMC. We employ this result within the proof of our characterization below.

**Lemma 1.** (i) $N$ and CDM imply CS. (ii) NC and DMC imply SC.

**Proof.** (i) Let $\varphi$ satisfy $N$ and CDM and let $i, j \in N$, $j \in \mathcal{P}$ (i) be symmetric in $(N, v)$. For all $K \subseteq N \setminus \{i, j\}$, we then have $v(K \cup \{i\}) - v(K \cup \{j\}) = 0 = 0(K \cup \{i\}) - 0(K \cup \{j\})$. By CDM, this implies $\varphi_i(N, v, P) - \varphi_j(N, v, P) = \varphi_i(N, 0, P) - \varphi_j(N, 0, P)$. Since all players are Null players in $(N, 0)$, the claim follows from $N$. Analogously, one shows (ii).

Now, we state and prove our main result. Note that the proof involves induction arguments which resemble the arguments used by Young (1985).

**Theorem 2.** The Owen value is the unique CS-value that satisfies $E$, $N$, NC, CDM, and DMC for fixed $N$ and $P$.

**Proof.** Fix $N$ and $\mathcal{P}$. For notational parsimony, we drop $N$ and $\mathcal{P}$ as arguments of CS-values throughout the proof. By (1), it is clear that the Owen value satisfies $E$, $N$, NC, CDM, and DMC. Let $\varphi$ be a CS-value that obeys all these axioms. By Lemma 1, $\varphi$ satisfies CS and SC.

For any $v \in V(N)$, set $T(v) := \{T \subseteq 2^N \setminus \{\emptyset\} \mid \lambda_T(v) \neq 0\}$, $\mathcal{P}(T(v)) := \{\mathcal{P}(T) \mid T \in T(v)\}$, and

$$v^{(D)} := \sum_{T \in T(v) : \mathcal{P}(T) = D} \lambda_T(v) \cdot u_T, \quad D \in \mathcal{P}(T(v)). \tag{2}$$

Note that either $P \subseteq D$ or $P \subseteq N \setminus D$ for all $P \in \mathcal{P}$ and $D \in \mathcal{P}(T(v))$.\[\square\]
By induction on $|P (T (v))|$, we first show that $\varphi (v) (P) = \text{Ow} (v) (P)$ for all $P \in \mathcal{P}$. If $|P (T (v))| = 0$, then $|T (v)| = 0$, i.e., $v = 0$ and therefore $\varphi (v) = \text{Ow} (v)$ by $\text{N}$. Let now $P (T (v)) = \{D\}$, $D \subseteq N$. If $P \in \mathcal{P}$, $P \subseteq N \setminus D$, then $P$ is a Null component in $(N, v, \mathcal{P})$. Hence, $\varphi (v) (P) = 0 = \text{Ow} (v) (P)$ by $\text{NC}$. Further, if $P, P' \in \mathcal{P}$ and $P, P' \subseteq D$, then $P$ and $P'$ are symmetric in $(N, v, \mathcal{P})$. Thus, $\varphi (v) (P) = \frac{v^{|N|}}{|P|} = \text{Ow} (v) (P)$ by $\text{E}$ and $\text{SC}$.

Assume now that $\varphi (v) (P) = \text{Ow} (v) (P)$ for all $P \in \mathcal{P}$ and all $v \in V (N)$ such that $|P (T (v))| \leq n$. Let $v \in V (N)$ be such that $|P (T (v))| = n + 1 > 1$. Consider $D \in P (T (v))$ and $P, P' \in \mathcal{P}$ such that $P, P' \subseteq D$ or $P, P' \subseteq N \setminus D$. By (2), we have

$$v (P (K) \cup P) - v (P (K) \cup P') = (v - v^{(D)}) (P (K) \cup P) - (v - v^{(D)}) (P (K) \cup P')$$

(3)

for all $K \subseteq N \setminus (P \cup P')$. Hence, $\text{DMC}$ implies

$$\varphi (v) (P) - \varphi (v) (P') = \varphi (v - v^{(D)}) (P) - \varphi (v - v^{(D)}) (P')$$

$$= \text{Ow} (v - v^{(D)}) - \text{Ow} (v - v^{(D)}) (P')$$

$$= \text{Ow} (v) (P) - \text{Ow} (v) (P')$$

where the second equation follows from the induction hypothesis and the third one holds by (3) and because $\text{Ow}$ meets $\text{DMC}$. Note that $|P (T (v - v^{(D)}))| = |P (T (v))| - 1$. We thus have

$$\Delta^v_P := \varphi (v) (P) - \text{Ow} (v) (P) = \varphi (v) (P') - \text{Ow} (v) (P') =: \Delta^v_{P'}$$

(4)

for $P, P' \subseteq D$ or $P, P' \subseteq N \setminus D$.

In the following, we show that (4) also holds for $P \subseteq D$ and $P' \subseteq N \setminus D$. Define

$$V^2 (N) := \{v \in V (N) | \exists \not= \emptyset \subseteq N : P (T (v)) = \{D, N \setminus D\}\}.$$ 

(5)

Case 1, $v \in V (N) \setminus V^2 (N)$: Since $|P (T (v))| > 1$, there are $D', D'' \in \mathcal{P} (T (v))$, $D' \neq D''$ such that $D' \cap D'' \neq \emptyset$ or $D' \cup D'' \neq N$. If (a) $D' \cap D'' \neq \emptyset$, then, w.l.o.g., there exist $P''$, $P''' \in \mathcal{P}$ such that $P'' \subseteq T' \cap T''$ and $P''' \subseteq T'' \setminus T'$. By the induction hypothesis, we have (4) and therefore $\Delta^v_P = \Delta^v_{P''} = \Delta^v_{P'''} = \Delta^v_{P'}$ for all $P \subseteq D'$ and $P' \subseteq N \setminus D'$. Together with (4), this implies $\Delta^v_P = \Delta^v_{P'}$ for all $P, P' \in \mathcal{P}$. If (b) $D' \cup D'' \neq N$, then, w.l.o.g., there are $P'' \subseteq D' \setminus D''$ and $P''' \subseteq N \setminus (D' \cup D'')$. Arguments as for (a) show that $\Delta^v_P = \Delta^v_{P'}$ for all $P, P' \in \mathcal{P}$.
Case 2, \( v \in V^2(N) \), i.e., \( \mathcal{P}(T(v)) = \{D, N\setminus D\} \), \( \emptyset \neq D \subsetneq N \). For \( P \subseteq D \) and \( P' \in N \setminus D \), consider the coalition function

\[
w = v^{(N\setminus D)} - \sum_{S \in T(v) : \mathcal{P}(S) = D} \lambda_S(v) \cdot u_{(S \cup P \setminus P')}.
\]

One easily checks that \( P, P', v, \) and \( w \) satisfy the hypothesis of DMC. Hence, we have

\[
\varphi(v)(P) - \varphi(v)(P') = \varphi(w)(P) - \varphi(w)(P'). \tag{6}
\]

Since \( \mathcal{P}(T(w)) = \{(D \setminus P) \cup P', N \setminus D\} \), \( w \in V(N \setminus V^2(N)) \). By \( |\mathcal{P}(T(w))| = 2 \) and by the induction basis, arguments as for Case 1 imply \( \Delta^v_P = \Delta^v_{P'} \). Thus, we have

\[
\begin{align*}
\varphi(v)(P) - \varphi(v)(P') &= \varphi(w)(P) - \varphi(w)(P') \\
&= \text{Ow}(w)(P) - \text{Ow}(w)(P') \\
&= \text{Ow}(v)(P) - \text{Ow}(v)(P'),
\end{align*}
\]

i.e., \( \Delta^v_P = \Delta^v_{P'} \), where the first equation is \([6]\), the second equation follows from \( \Delta^v_P = \Delta^v_{P'} \), and third equation holds because Ow meets DMC. Together with \([4]\) and the induction basis, this implies \( \Delta^v_P = \Delta^v_{P'} \) for all \( P, P' \in \mathcal{P} \).

Since the Cases 1–2 are exhaustive, we have \( \Delta^v_P = \Delta^v_{P'} \) for all \( P, P' \in \mathcal{P} \) and \( v \in V(N) \). Summing up \([4]\) over \( P' \in \mathcal{P} \) yields

\[
|\mathcal{P}|(\varphi(v)(P) - \text{Ow}(v)(P)) = \varphi(v)(N) - \text{Ow}(v)(N) \equiv 0,
\]

hence

\[
\varphi(v)(P) = \text{Ow}(v)(P), \quad P \in \mathcal{P}, \quad v \in V(N), \tag{7}
\]

i.e., \( \varphi \) distributes \( v(N) \) among the components in the same way as the Owen value.

We now turn to the distribution of payoffs within the components. Fix some \( P \in \mathcal{P} \) and define

\[
T(v, P) := \{T \cap P | T \in T(v) : T \cap P \neq \emptyset\}.
\]

If \( |T(v, P)| = 0 \), then all players in \( P \) are Null players in \( (N, v) \). If \( |T(v, P)| = 1 \), i.e., \( T(v, P) = \{K\} \), then all players in \( P \setminus K \) are Null players in \( (N, v) \) and the players in \( K \) are symmetric in \( (N, v) \). Hence, \([7]\), CS, and N imply \( \varphi_i(v) = \text{Ow}_i(v) \) for all \( i \in P \).

One continues by induction on \( |T(v, P)| \). Since the full argument, even the wording is almost the same as above, we leave it to the reader. Some remarks indicate
how to proceed. The players $i \in P$ play the role of the components $P \in \mathcal{P}$ and $T(v, P)$ replaces $\mathcal{P}(T(v))$. Further, one employs the coalition functions $v^{(T)} = \sum_{S \in \mathcal{T}(v) : S \cap P = T} \lambda_S(v) \cdot u_S$, $T \in \mathcal{T}(v, P)$ instead of (2). Finally, one uses CDM and (7) instead of DMC and E, respectively. □

Remark 1. It is well known/easy to check that the AD-value (Aumann and Drèze, 1974) satisfies $N$, CDM, and component efficiency (CE), i.e., $\varphi(N, v, \mathcal{P})(P) = v(P)$ for all $P \in \mathcal{P}$. To show that the AD-value is the unique such value, one may modify the above proof as follows. By CE, the first part of the proof concerning the distribution of $v(N)$ between the components can be dropped. Using $\varphi(N, v, \mathcal{P})(P) = v(P)$, one follows the hints in the second part of the proof concerning the distribution of $\varphi(N, v, \mathcal{P})(P)$ within $P$.

Remark 2. Owen (1981) suggests the following extension of the Banzhaf value (Banzhaf, 1965; Owen, 1975) to CS-games,

$$BO_i(N, v, \mathcal{P}) = \frac{1}{2^{|\mathcal{P}|-1}} \frac{1}{2^{|\mathcal{P}(i)|-1}} \sum_{c \subseteq \mathcal{P}\setminus\{\mathcal{P}(i)\}} \sum_{K \subseteq \mathcal{P}(v)\setminus\{i\}} MC_i^v \left( K \cup \bigcup_{P \in c} P \right), \quad i \in N.$$ (8)

From (8), it is immediate that BO obeys N as well as CDM. However, BO neither satisfies DMC nor NC. Consider $N = \{1, 2, 3, 4, 5\}$, $\mathcal{P} = \{\{1, 2, 3\}, \{4, 5\}\}$, $v = u_N$, and $w = u_{\{1,4\}}$. This gives

$$BO(N, v, \mathcal{P}) = \left( \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{4} \right) \quad \text{and} \quad BO(N, w, \mathcal{P}) = \left( \frac{1}{2^2}, 0, 0, \frac{1}{2^2}, 0 \right).$$

Further, it is easy to check that $v$, $w$, $\{1, 2, 3\}$, and $\{4, 5\}$ satisfy the hypothesis of DMC. Yet,

$$BO(N, v, \mathcal{P})(\{1, 2, 3\}) - BO(N, v, \mathcal{P})(\{4, 5\}) = \frac{3}{8} - \frac{1}{2} \neq \frac{1}{2} - \frac{1}{2} = BO(N, w, \mathcal{P})(\{1, 2, 3\}) - BO(N, w, \mathcal{P})(\{4, 5\}),$$

i.e., DMC is violated. Consider now $N = \{1, 2, 3, 4\}$, $\mathcal{P} = \{N\}$, $v = u_{\{1\}} - u_{\{2,3,4\}}$. This gives

$$BO(N, v, \mathcal{P}) = \left( 1, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4} \right),$$

hence $BO(N, v, \mathcal{P})(N) = \frac{1}{4}$. Since $N$ is a Null component, NC is violated.
3.3. Non-redundancy. Our axiomatization is non-redundant. To show this, we provide CS-values which satisfy any four of our axioms for all pairs \((N, P)\), but not the remaining one for some pair \((N, P)\). Some arguments below involve the following lemma.

Lemma 3. (i) A and CS imply CDM. (ii) A and SC imply DMC.

Proof. (i) Let \(\varphi\) satisfy A and CS, and let \(i, j \in N, P\), and \(v, w \in V(N)\) satisfy the hypothesis of CDM. Then, \(i\) and \(j\) are symmetric in \((N, v - w)\), and we have

\[
\varphi_i(N, v, P) - \varphi_j(N, v, P) = \varphi_i(N, w, P) - \varphi_j(N, w, P)
\]

\[
+ \varphi_i(N, v - w, P) - \varphi_j(N, v - w, P)
\]

\[
\overset{\text{CS}}{=} \varphi_i(N, w, P) - \varphi_j(N, w, P),
\]

i.e., \(\varphi\) satisfies CDM. Analogously, one shows (ii).

Example 1. The CS-value \(\varphi^{(1)}\) given by \(\varphi^{(1)}_i(N, v, P) = 0, i \in N\) satisfies N, NC, CDM and DMC, but not E, in general.

Example 2. The CS-value \(\varphi^{(2)}\) given by \(\varphi^{(2)}_i(N, v, P) = \frac{\Omega_w(N, v, P)(P(i))}{|P(v)|}, i \in N\) satisfies E, NC, CDM and DMC, but not N, in general. The CS-value \(\varphi^{(2)}\) inherits E, NC, and A from Ow. Since it is easy to check that it obeys CS and SC, CDM and DMC follow from Lemma 3.

Example 3. Let \(N_0(v)\) denote the set of Null players in \((N, v)\) and let \(P_0(v)\) denote the set of Null components in \((N, v, P)\). Define \(P_{00}(v) := \{P \in P| P \subseteq N_0(v)\}\).

The CS-value \(\varphi^{(3)}\) given by

\[
\varphi^{(3)}_i(N, v, P) = \begin{cases} 
0, & i \in N_0(v), \\
0, & i \notin N_0(v), P_{00}(v) \neq \emptyset, P(i) \in P_0(v), \\
v(N), & i \notin N_0(v), P_{00}(v) \neq \emptyset, P(i) \notin P_0(v), \\
\frac{|P \setminus P_0(v) \setminus P(i) \setminus N_0(v)|}{|P|}, & i \notin N_0(v), P_{00}(v) = \emptyset, \\
\frac{|P \setminus N_0(v)|}{|P|}, & i \notin N_0(v), P_{00}(v) = \emptyset,
\end{cases}
\]

\(i \in N\), satisfies E, N, CDM and DMC, but not NC, in general. It is easy to check that \(\varphi^{(3)}\) meets E and N. The last case ensures that \(\varphi^{(3)}\) fails NC for some \(v \in V(N)\). To see CDM and DMC, observe that all non-Null players within the same component obtain the same payoff and that all non-Null components also obtain the same payoffs.
Example 4. Fix some $\sigma_p \in \Sigma(P)$ for all $P \in \mathcal{P}$ and set
\[
\Sigma^* := \{ \sigma \in \Sigma(N,\mathcal{P}) | \forall P \in \mathcal{P}, i,j \in P : \sigma(i) - \sigma(j) = \sigma_P(i) - \sigma_P(j) \}.
\]
The CS-value $\varphi^{(4)}$ given by $\varphi^{(4)}_i(N,v;\mathcal{P}) = \frac{1}{|\Sigma^*|} \sum_{\sigma \in \Sigma^*} MC_i^v(\sigma), i \in N$ satisfies $E$, $N$, $NC$, and $DMC$, but not $CDM$, in general. $E$, $N$, and $NC$ are immediate. Further, it is easy to check that $\varphi^{(4)}$ satisfies $A$ and $SC$, but not $CS$, in general. By Lemma 3(i), $\neg CS$ and $N$ imply $\neg CDM$; and by Lemma 3(ii), $A$ and $SC$ imply $DMC$.

Example 5. Fix some $\rho \in \Sigma(P)$ and set
\[
\Sigma^{**} := \{ \sigma \in \Sigma(N,\mathcal{P}) | \forall i,j \in N : \rho(\mathcal{P}(i)) \leq \rho(\mathcal{P}(j)) \iff \sigma(i) \leq \sigma(j) \}.
\]
The CS-value $\varphi^{(5)}$ given by $\varphi^{(5)}_i(N,v;\mathcal{P}) = \frac{1}{|\Sigma^{**}|} \sum_{\sigma \in \Sigma^{**}} MC_i^v(\sigma), i \in N$ satisfies $E$, $N$, $NC$, and $CDM$, but not $DMC$, in general. $E$, $N$, and $NC$ are immediate. Further, it is easy to check that $\varphi^{(5)}$ satisfies $A$ and $CS$, but not $SC$, in general. By Lemma 3(ii), $\neg SC$ and $NC$ imply $\neg DMC$; and by Lemma 3(ii), $A$ and $CS$ imply $CDM$.

3.4. Relation to marginality. Since $M$ on the one hand and $CDM$ and $DMC$ on the other hand are relatives, one might be suspicious whether these axioms can be inferred from each other in a simple way, i.e., not via the fact that these axioms are part of characterizations of the Owen value. By help of some examples, we show that this is not the case.

Example 6. $CDM$, $DMC$, $N$, $NC$ and do not imply $M$, in general. Consider the CS-value $\varphi^{(6)}$ given by
\[
\varphi^{(6)}_i(N,v;\mathcal{P}) = \frac{\prod_{T \in 2^N \setminus \{\emptyset\}} v(T)}{|\mathcal{P}| \cdot |\mathcal{P}(i)|}, \quad i \in N.
\]
Since $\varphi^{(6)}(N,v;\mathcal{P}) = \varphi^{(6)}_j(N,v;\mathcal{P})$ for $j \in \mathcal{P}(i)$, $\varphi^{(6)}$ satisfies $CDM$, and since $\varphi^{(6)}(N,v;\mathcal{P})(P) = \varphi^{(6)}(N,v;\mathcal{P})(P')$ for $P,P' \in \mathcal{P}$, $\varphi^{(6)}$ also satisfies $DMC$. If $i$ is a Null player in $(N,v)$, then $v(\{i\}) = 0$. Hence, $\varphi^{(6)}$ obeys $N$. If $P$ is a Null component in $(N,v;\mathcal{P})$, then $v(P) = 0$. Hence, $\varphi^{(6)}$ obeys $NC$. Consider now $N = \{1,2\}$, $\mathcal{P} = \{N\}$, and $v = u_{\{1\}} + u_{\{2\}}$ and $w = u_{\{1\}}$. It is easy to check that $1$, $v$, and $w$ satisfy the hypothesis of $M$. However, $\varphi^{(6)}_1(N,v;\mathcal{P}) = 1 \neq 0 = \varphi^{(6)}_1(N,v;\mathcal{P})$, i.e., $\varphi^{(6)}$ does not satisfy $M$, in general.
Example 7. M, CS, and SC do not imply CDM or DMC, in general. Consider the CS-value \( \varphi^{(7)} \) given by
\[
\varphi^{(7)}_i (N, v, \mathcal{P}) = (\text{Ow}_i (N, v, \mathcal{P}))^3, \quad i \in N.
\]
Obviously, \( \varphi^{(7)} \) inherits M, CS, and SC from Ow. Consider now \( N = \{1, 2\} \), \( \mathcal{P} = \{N\} \), \( v = 2 \cdot u_{\{1\}} + u_{\{2\}} \), and \( w = u_{\{1\}} \). It is easy to check that 1, 2, \( v \), and \( w \) satisfy the hypothesis of CDM. However, \( \varphi^{(7)}_1 (N, v, \mathcal{P}) - \varphi^{(7)}_2 (N, v, \mathcal{P}) = 2^3 - 1^3 \neq 1^3 - 0^3 = \varphi^{(7)}_1 (N, w, \mathcal{P}) - \varphi^{(7)}_2 (N, w, \mathcal{P}) \), i.e., \( \varphi^{(7)} \) does not satisfy CDM. Using \( \mathcal{P}' = \{\{1\}, \{2\}\} \), one shows that \( \varphi^{(7)} \) does not satisfy DMC.

Example 8. CDM, DMC, and E do not imply M, in general. Consider the CS-value \( \varphi^{(8)} \) given by \( \varphi^{(8)}_i (N, v, \mathcal{P}) = \frac{v(N)}{|P_{i}\cap\mathcal{P}|} \), \( i \in N \). Obviously, \( \varphi^{(8)} \) satisfies CDM, DMC, and E. Consider now \( N = \{1, 2\} \), \( \mathcal{P} = \{N\} \), \( v = u_{\{1\}} + u_{\{2\}} \) and \( w = u_{\{1\}} \). It is easy to check that 1, \( v \), and \( w \) satisfy the hypothesis of M. However, \( \varphi^{(8)}_1 (N, v, \mathcal{P}) = 1 \neq \frac{1}{2} = \varphi^{(8)}_2 (N, v, \mathcal{P}) \), i.e., \( \varphi^{(8)} \) does not satisfy M.

Example 9. M and E do not imply CDM or DMC, in general. To see this, observe that the CS-values \( \varphi^{(4)} \) and \( \varphi^{(5)} \) from Examples 4 and 2 both satisfy M.

Despite of the independence of M and CDM or DMC, Khmelnitskaya and Yannakakis (2007, Theorem 1) entails a characterization of the Owen value involving CDM and DMC. In particular, one obtains such a characterization from Theorem 2 by replacing N with a CS-value version of the Null-player-out axiom (see e.g. Derks and Haller, 1999) below. However, this axiomatization is much stronger than the original one. First, NPO relates games with different player sets and has implications on the payoffs of a multitude of players. Second, N can easily be inferred from E and NPO (see the proof of Corollary 3 below).

Null player out, NPO. If \( i \in N \) is a Null player in \( (N, v) \), then \( \varphi_j (N, v, \mathcal{P}) = \varphi_j (N \setminus \{i\}, v|_{N\setminus\{i\}}, \mathcal{P}|_{N\setminus\{i\}}) \) for all \( j \in N \setminus \{i\} \).

The following lemma establishes a relation between CDM and M, which is employed in a proof below.

Lemma 4. N, NPO, and CDM imply M.

Proof. Let \( \varphi \) satisfy N, NPO, and CDM, let \( \mathcal{P} \) be a coalition structure on \( N \), and let \( i \in N \) and \( v, w \in V(N) \) satisfy the hypothesis of M, i.e.,
\[
v(K \cup \{i\}) - v(K) = w(K \cup \{i\}) - w(K), \quad K \subseteq N \setminus \{i\}.
\]
Consider some player $\theta \notin N$ and set $\bar{N} := N \cup \{\theta\}$, $\bar{v}, \bar{w} \in V(\bar{N})$, $\bar{v}(K) := v(K \setminus \{\theta\})$, $\bar{w}(K) := w(K \setminus \{\theta\})$ for all $K \subseteq \bar{N}$, and a coalition structure $\bar{\mathcal{P}}$ on $\bar{N}$ such that $\bar{\mathcal{P}}(j) := \mathcal{P}(j)$ for all $j \in N \setminus \mathcal{P}(i)$, $\bar{\mathcal{P}}(j) := \mathcal{P}(i) \cup \{\theta\}$ for all $j \in \mathcal{P}(i)$. Since $\theta$ is a Null player in $(\bar{N}, \bar{v})$ and in $(\bar{N}, \bar{w})$, (9) implies $\bar{v}(K \cup \{i\}) - \bar{v}(K \setminus \{\theta\}) = \bar{w}(K \cup \{i\}) - \bar{w}(K \setminus \{\theta\})$ for all $K \subseteq \bar{N} \setminus \{i, \theta\}$. By $\theta \in \bar{\mathcal{P}}(i)$ and using CDM, we conclude $\varphi_i(\bar{N}, \bar{v}, \bar{\mathcal{P}}) = \varphi_i(\bar{N}, \bar{w}, \bar{\mathcal{P}})$ by $\textbf{N}$. Since $N = \bar{N} \setminus \{\theta\}$, $\mathcal{P}|_N = \mathcal{P}$, $\bar{v}|_N = v$, and $\bar{w}|_N = w$, $\textbf{NPO}$ implies $\varphi_i(N, v, \mathcal{P}) = \varphi_i(N, w, \mathcal{P})$, which proves the claim. 

Corollary 5. The Owen value is the unique CS-value that satisfies $\textbf{E}$, $\textbf{NPO}$, $\textbf{NC}$, CDM and DMC.

Proof. By [1], the Owen value obeys all these axioms. Let $\varphi$ satisfy $\textbf{E}$, $\textbf{NPO}$, $\textbf{NC}$, CDM and DMC. If $i$ is a Null player in $(N, v)$, then $v(N) = v(N \setminus \{i\})$ and therefore

$$\varphi_i(N, v, \mathcal{P}) = E \quad v(N) - \sum_{j \in N \setminus \{i\}} \varphi_j(N, v, \mathcal{P})$$

$$\textbf{NPO} \quad v(N \setminus \{i\}) - \sum_{j \in N \setminus \{i\}} \varphi_j(N \setminus \{i\}, v|_{N \setminus \{i\}}, \mathcal{P}|_{N \setminus \{i\}}) = 0.$$

Hence, $\textbf{E}$ and $\textbf{NPO}$ imply $\textbf{N}$. By Lemma [1], $\varphi$ also satisfies CS and SC. Finally, Lemma [4] implies that $\varphi$ obeys $\textbf{M}$. Thus by Khmelnitskaya and Yanovskaya (2007, Theorem 1), $\varphi$ is unique. 

4. Discussion

We now indicate how our characterization of the Owen value can be adapted for the Owen value for games with a level structure (LS-games) (Winter, 1989). A level structure on $N$ is a tuple $\mathcal{L} = (\mathcal{P}_1, \ldots, \mathcal{P}_p)$, $p \in \mathbb{N}$ of coalition structures on $N$ such that $\mathcal{P}_{k+1}$ is finer than $\mathcal{P}_k$, i.e., $\mathcal{P}_{k+1}(i) \subseteq \mathcal{P}_k(i)$ for all $i \in N$ and $k \in \{1, \ldots, p-1\}$. An LS-value $\varphi$ for $N$ and $\mathcal{L}$ assigns a payoff vector $\varphi(N, v, \mathcal{L}) \in \mathbb{R}^N$ to all LS-games.

We modify NC, CDM, and DMC as follows:

Null level component, NLC. If $P \in \mathcal{P}_k$, $k \in \{1, \ldots, p\}$ is a Null component in $(N, v, \mathcal{P}_k)$ then

$$\sum_{i \in P} \varphi_i(N, v, \mathcal{L}) = 0.$$
Differential marginality within level components, LCDM. If \( j \in \mathcal{P}_p \{i\} \) and
\[
v(K \cup \{i\}) - v(K \cup \{j\}) = w(K \cup \{i\}) - w(K \cup \{j\})
\]
for all \( K \subseteq N \setminus \{i,j\} \), then
\[
\varphi_i(N,v,L) - \varphi_j(N,v,L) = \varphi_i(N,w,L) - \varphi_j(N,w,L).
\]

Differential marginality between level components, DMLC. Set \( P_0 = \{N\} \).
If \( P, P' \in \mathcal{P}_k, k \in \{1, \ldots, p\} \), \( P' \subseteq \mathcal{P}_{k-1}(P) \), and
\[
v(P_k(K) \cup P) - v(P_k(K) \cup P') = w(P_k(K) \cup P) - w(P_k(K) \cup P')
\]
for all \( K \subseteq N \setminus (P \cup P') \), then
\[
\sum_{i \in P} \varphi_i(N,v,L) - \sum_{i \in P'} \varphi_i(N,v,L) = \sum_{i \in P} \varphi_i(N,w,L) - \sum_{i \in P'} \varphi_i(N,w,L).
\]

The following theorem can be established by applying the arguments from the proof of Theorem 2 repeatedly to all levels top down.

Theorem 6. The Winter value is the unique LS-value that satisfies \( E, N, NLC, LCDM \), and \( DMLC \) for fixed \( N \) and \( L \).

Remark 3. There is some analogon to Remark 7 for LS-games. Instead of \( E \), Winter’s second LS-value satisfies top level component efficiency (TCE), i.e., \( \varphi(N,v,L)(P) = v(P) \) for all \( P \in \mathcal{P}_1 \). Using the techniques from the proof of Theorem 2 and the hints in Remark 7, one shows that this LS-value is characterized by \( TCE, N, NLC, LCDM \), and \( DMLC \).

References


